The Final Set of Meromorphic And Entire Functions

by

Aiman Mustafa Mahmoud Gusti

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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under the direction of his thesis committee, and approved by all its members, has been presented to and accepted by the Dean, College of Graduate Studies, in partial fulfillment of the requirements for the degree of

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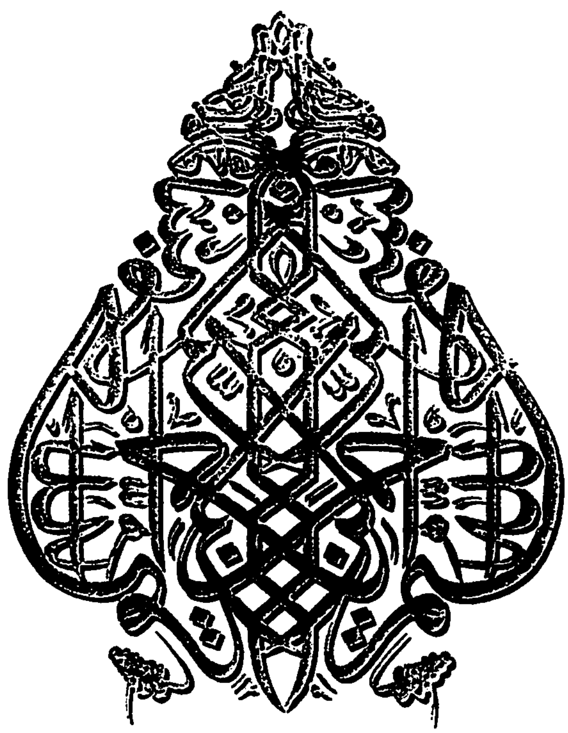

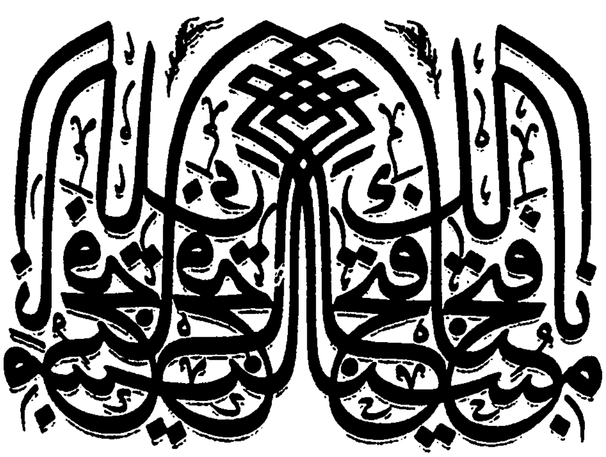

To the Sun of Hope and Source of Giving MY MOTHER

To the Emblem of Sacrifice and Symbol of Strong Will MY FATHER

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Praise to God, Lord of the worlds, the Almighty, with whose gracious help it was possible to accomplish this thesis.

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الفئة النهائية للدوال الميرومورفية سوف تعرّف بالتفصيل ، بالاضافة الى اثبات نظرية بوليا اثباتا تحليليا لتحديد الفئة النهائية لأي دالــــــه ميرومورفية . سوف يكون هناك أيضا دراسة تفصيلية لاصفار بعض كثيرات الحدود لكي نحدد موقع اصفار بعض الدوال الميرومورفية والكاملة التحليل . أخيرا ، سوف يتم تحليل سريع على أصفار المشتقات المتتابعة لدالة كاملة التحليــــل باستخدام بعض الشروط المتعلقة بالدالة ، بالاضافة الى اعطاء نظرة مختصــرة على أحدث الدراسات المتعلقة بموضوع اصفار الدوال الكاملة التحليل .

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THESIS ABSTRACT

Full Name Of Student : **Aiman Mustafa Mahmoud Gusti**

Title Of Study : **The Final Set Of Meromorphic And Entire Functions**

Major Field : **Mathematics**

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The final set of a meromorphic function is introduced and a theorem of Pólya is proved in detail to determine such sets for any meromorphic function. A close study of the zeros of some classical polynomials is given in order to determine the zeros of the successive derivatives of some meromorphic and entire functions. Also a quick analysis of the zeros of the successive derivatives of an entire function using some restrictions on the function and its zeros is done, followed by a brief look on the most recent developments on the zero sets of entire functions.

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INTRODUCTION

According to Rolle's theorem, the derivative of a differentiable real function f vanishes at least once between any two zeros of f. For complex valued functions this is no more true. For example $f(z) = e^{iz} - 1$ has an infinite number of real zeros but its derivative has no zeros. Another function $f(z) = (z-1)e^{z^2}$ shows that even if f' has zeros they do not have to lie between the zeros of f. On the other hand, Lucas' theorem tells us that the zeros of the derivative of a polynomial lie in the smallest convex polygon that contains the zeros of the polynomial. Both Rolle's theorem and Lucas' theorem may be viewed as supplying information about the location of the zeros of the derivative of a function in relation to the location of the zeros of the function. It is then natural to ask about the effect of differentiation of a general function. More precisely if f is analytic, does differentiation of f introduce zeros? If it does, where can these zeros be located? What about successive derivatives? In the words of Pólya: "How will the behavior of the zeros of the nth derivative of f be when n is very large? Does this behavior depend on the analytic nature of the function f? Will the zeros of the successive derivatives of f have any definite trend? If they do, can we find this trend?"

We shall see in this thesis that the behavior of the zeros of the successive derivatives of $f(z)$ does depend on the analytic nature of the function. In fact, the final position of these zeros, which we call later the final set (defined in Chapter 1), can be easily recognized for any meromorphic function using the main theorem of Pólya on meromorphic functions.

In Chapter 1, we study the example $f(z) = \frac{2}{z^2-1}$ and show by elementary methods that its final set is the perpendicular bisector of the line segment joining its two poles ± 1. In the general case it turns out that the poles of a meromorphic function $f(z)$ acts as repulsive centers on the zeros of the successive derivatives of f. This is the main theorem of Pólya whose proof is given in Chapter 2 together with some applications.

Since no analogue of Pólya's theorem exists for entire functions, we then consider the question of the existence of zeros for the successive derivatives of an entire function. In Chapter 3 and 4 we determine the set of zeros of successive derivatives of certain entire functions. It turns out that there is a close relationship between the derivatives of these functions and certain polynomials some of which are classical polynomials. This makes it possible to determine precisely the location of zeros.

In the last chapter we will see that if an entire function has only real zeros then the successive derivatives of this function may have non-real zeros. We shall also see how the genus and order of entire functions affect the trend of the zeros of the successive derivatives of these functions. Finally, we shall introduce some of the most recent developments on the zero sets of entire functions.

Chapter 1

Introduction To the Final Set

In this chapter we define the notion of the final set of a meromorphic function and give a detailed example where we find the final set of a specific function using only the definition.

1.1 The Final Set

Definition 1.1.1 *The final set of a meromorphic function f consists of all points z of the complex plane where any circle (neighborhood) of center (around) z, contains zeros of infinitely many among the successive derivatives $f'(z), f''(z), f'''(z), \cdots$ of $f(z)$. [11; 478]*

In other words, if E is the union of all the zero sets of the successive derivatives of $f(z)$, then the final set of f is the set of limit points of E.

Thus in order to find the final set of any function using this definition, we must first find the zero set of each derivative of $f(z)$ and find their union, then we find the limit points of this union. These limit points constitute the final set of the given function.

This procedure will be used in the next section to find the final set of the meromorphic function $\frac{2}{z^2-1}$.

1.2 The Final Set of the Function $f(z) = \frac{2}{z^2-1}$.

In addition to the above definition, the following two propositions will be used to find the final set of the meromorphic function $f(z) = \frac{2}{z^2-1}$.

Proposition 1.2.1 *The linear fractional transformation* $T(z) = \frac{1+z}{1-z}$ *takes the set* $\{|z| = 1, z \neq 1\}$ *in a 1-1 fashion onto the y-axis, where the inverse of* T *has the form* $T^{-1}(w) = \frac{1-w}{1+w}$.

Proof: Let us consider first some points on the unit circle and find their images under the transformation T. We can easily see that $T(1) = \infty, T(i) = i$ and $T(-1) = 0$. So, by the circle preserving property of linear fractional transformations, the points ∞, i and 0 must be on a circle or a straight line induced by transforming the unit circle under T. Also $T(0) = 1$ and $T(\infty) = -1$. But 0 and ∞ are symmetric with respect to the unit circle. Therefore, by the symmetric property of linear fractional transformation, 1 and -1 are symmetric with respect to the induced circle.

Therefore, and due to the fact that any circle can be determined by three points, the induced circle is the y-axis.

On the other hand, since $T(z)$ is a linear fractional transformation it must be a 1-1 [7;274], and thus have an inverse equals to $T^{-1}(w) = \frac{1-w}{1+w}$.

Hence $T(z)$ takes the unit circle in a 1-1 fashion onto the y-axis.

Proposition 1.2.2 *The limit points of the set*

$$E = \bigcup_{n=1}^{\infty} \left\{ \frac{1+e^{\frac{2\pi i}{(n+1)}}}{1-e^{\frac{2\pi i}{(n+1)}}}, \frac{1+e^{\frac{4\pi i}{(n+1)}}}{1-e^{\frac{4\pi i}{(n+1)}}}, \ldots, \frac{1+e^{\frac{2n\pi i}{(n+1)}}}{1-e^{\frac{2n\pi i}{(n+1)}}} \right\}$$

is the y-axis.

Proof: To prove this proposition we shall show that the set of all limit points of E is a subset of the y-axis and conversely.

We first show that the set of limit points of E is a subset of the y-axis. So let z be a point outside the y-axis. Then $z = x + iy, x \neq 0$. Also let $\delta = \frac{|x|}{2}$, and let $B(z, \delta)$ be the open disk of center z and radius δ, see figure (1.1).

Since any element of E is either zero or of the form $\frac{1 \pm w}{1 - w}$ where $|w| = 1$, then, by the previous proposition, E is a subset of the y-axis. Thus, $B(z, \delta) \cap E = \phi$, i.e., the point z is not a limit point of E. Therefore, any point outside the y-axis is not a limit point of E. Thus the limit points of E are contained in the y-axis.

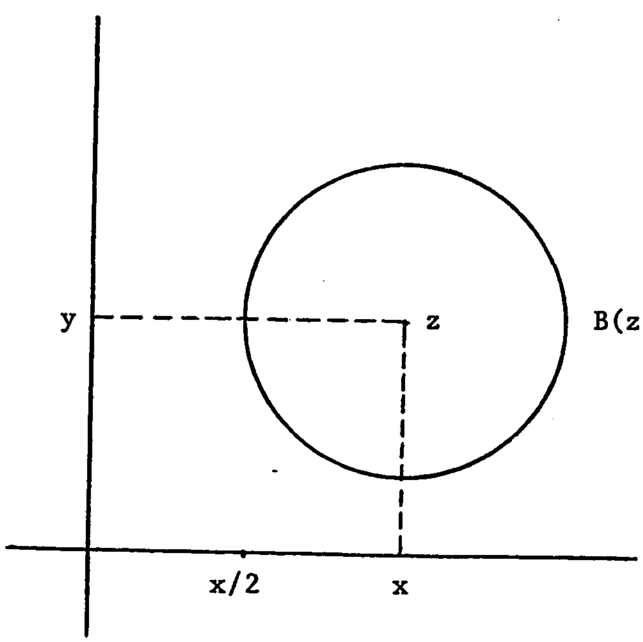

Figure (1.1)

Conversely, let z be a point on the y-axis, i.e., $z = iy, y \in \Re$, let $\epsilon > 0$ be given and write $(i(y-\epsilon), i(y+\epsilon)) = B(z,\epsilon) \cap \{Rez = 0\}$. That is $(i(y-\epsilon), i(y+\epsilon))$ is an interval along the y-axis centered at iy. Since any linear fractional transformation has an inverse , then $T(z) = \frac{1+z}{1-z}$ also has an inverse that equals $T^{-1}(w) = \frac{1-w}{1+w}$ which is continuous, in fact analytic except at $w = -1$. Therefore, the image of the connected subset $(i(y-\epsilon), i(y+\epsilon))$ of the y-axis under the continuous mapping T^{-1} is a connected subset of the unit circle (since the image of a connected subset of a topological space under a continuous map is connected). But, a connected subset of any circle is an arc of that circle. Thus we can denote the induced connected subset of the unit circle by an arc of the form $(e^{i\theta_1}, e^{i\theta_2}), 0 < \theta_1 < \theta_2 < 2\pi$.

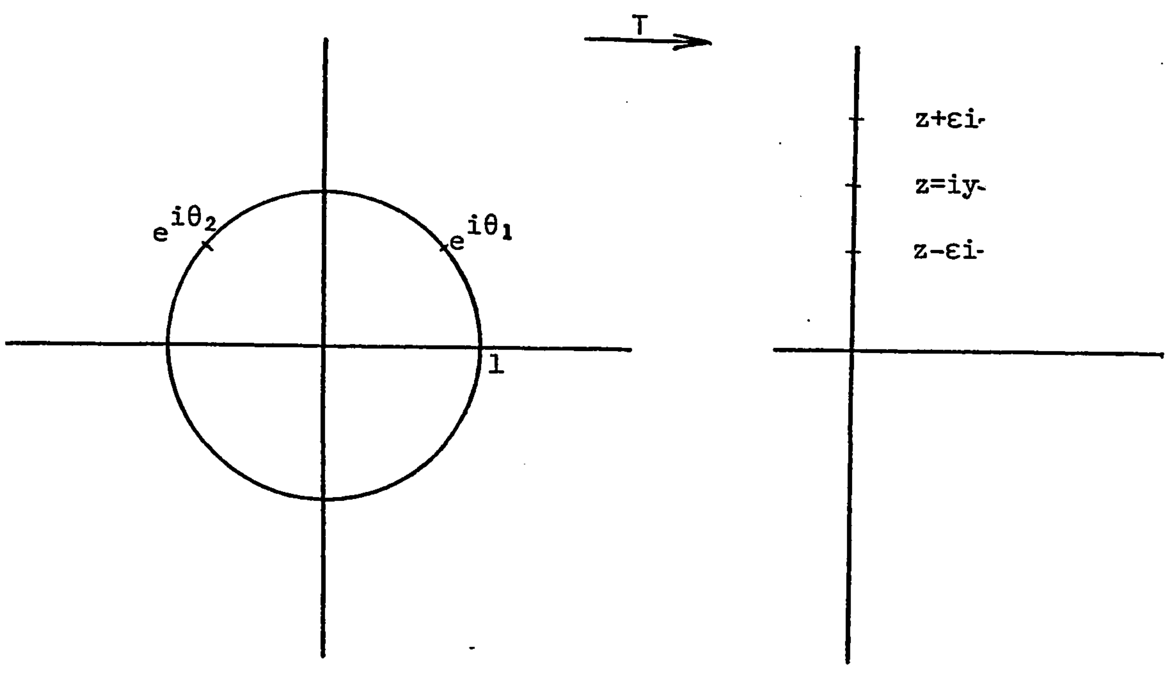

Figure (1.2)

Since the rational numbers are dense in the real numbers $\Re$, one can find a rational number r such that $0 < \frac{\theta_1}{2\pi} < r < \frac{\theta_2}{2\pi} < 1$. Thus there exists positive integers k and n such that

$$0 < \frac{\theta_1}{2\pi} < \frac{k}{n+1} < \frac{\theta_2}{2\pi} < 1,$$

or

$$0 < \theta_1 < \frac{2k\pi}{n+1} < \theta_2 < 2\pi.$$

Hence

$$e^{\frac{2k\pi i}{n+1}} \in (e^{i\theta_1}, e^{i\theta_2}) = T^{-1}(i(y-\epsilon), i(y+\epsilon)).$$

If $T(e^{\frac{2k\pi i}{n+1}}) = iy$ we may choose another rational number r and corresponding integers k and n such that $T(e^{\frac{2k\pi i}{n+1}}) \neq iy$. This is possible since T is one-to-one. Therefore $T(e^{\frac{2k\pi i}{n+1}}) \in (i(y-\epsilon), i(y+\epsilon)) - \{iy\}$. But $T(e^{\frac{2k\pi i}{n+1}}) \in E$, since $T(e^{\frac{2k\pi i}{n+1}}) = \frac{1+e^{\frac{2k\pi i}{n+1}}}{1-e^{\frac{2k\pi i}{n+1}}}$.

Thus, $(z - i\epsilon, z + i\epsilon) \cap E \neq \phi$, i.e., $(B(z,\epsilon) \setminus \{z\}) \cap E \neq \phi$. So z is a limit point of E. Hence any point of the y-axis is a limit point of E, i.e., the y-axis is a subset of the limit points of E. Therefore the set of limit points of E is the y-axis.

We shall now use the above two propositions to find the final set of the desired meromorphic function $f(z) = \frac{2}{z^2-1}$.

Theorem 1.2.1 *The final set of the meromorphic function $f(z) = \frac{2}{z^2-1}$ is the set of all points of the y-axis.*

Proof: We will now use the definition of the final set and follow it step-by-step to prove the theorem.

Let us first find the zero sets of all successive derivatives of $f(z)$. Note that the function $f(z)$ can also be written, by partial fractions, as

$$f(z) = \frac{1}{z-1} - \frac{1}{z+1}.$$

Therefore, the nth derivative of $f(z)$ will be

$$f^{(n)}(z) = \frac{(-1)^n n!}{(z-1)^{n+1}} - \frac{(-1)^n n!}{(z+1)^{n+1}} = (-1)^n n! \left\{ (\frac{1}{z-1})^{n+1} - (\frac{1}{z+1})^{n+1} \right\}.$$

To find the zero set of $f^{(n)}(z)$, let $f^{(n)}(z) = 0$. Then

$$(-1)^n n! \left\{ (\frac{1}{z-1})^{n+1} - (\frac{1}{z+1})^{n+1} \right\} = 0$$

or

$$(\frac{1}{z-1})^{n+1} = (\frac{1}{z+1})^{n+1}$$

or

$$\left[\frac{(z-1)}{(z+1)} \right]^{n+1} = 1.$$

Thus

$$\frac{(z-1)}{(z+1)} = 1, e^{\frac{2\pi i}{(n+1)}}, ..., e^{\frac{2n\pi i}{(n+1)}}.$$

But if $\frac{(z-1)}{(z+1)} = 1$, then $1 = -1$ which is absurd. Thus $\frac{z-1}{z+1}$ can not equal to 1. Hence

$$\frac{(z-1)}{(z+1)} = e^{\frac{2\pi i}{(n+1)}}, \cdots, e^{\frac{2n\pi i}{(n+1)}}$$

or

$$z = \frac{1 + e^{\frac{2\pi i}{(n+1)}}}{1 - e^{\frac{2\pi i}{(n+1)}}}, \cdots, \frac{1 + e^{\frac{2n\pi i}{(n+1)}}}{1 - e^{\frac{2n\pi i}{(n+1)}}}.$$

Therefore the zero set of $f^{(n)}(z)$ is

$$A_n = \left\{ \frac{1 + e^{\frac{2k\pi i}{n+1}}}{1 - e^{\frac{2k\pi i}{n+1}}} : k = 1, 2, \cdots, n \right\}$$

where $n = 1, 2, 3, \cdots$.

After we have found the zero sets of all the derivatives of $f(z)$ let E be the union of all these sets, i.e.

$$E = \bigcup_{n=1}^{\infty} A_n$$

or

$$E = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left\{ \frac{1 + e^{\frac{2k\pi i}{n+1}}}{1 - e^{\frac{2k\pi i}{n+1}}} : k = 1, 2, \cdots, n \right\}.$$

It now becomes apparent that all the points of E are of the form $\frac{1+w}{1-w}$, where w is on the unit circle. Therefore they must lie on the y-axis since the transformation $T(w) = \frac{1+w}{1-w}$ takes the unit circle onto the y-axis, by proposition (1.2.1).

Also, by the previous proposition, the limit points of E is the set of all points of the y-axis. Thus, by the definition of the final set, the final set of the function $f(z) = \frac{2}{z^2 - 1}$ is the set of all points of the y-axis.

Chapter 2

Pólya's Theorem On Meromorphic Functions

Note that the final set of $f(z) = \frac{2}{z^2 - 1}$ turned out to be the y-axis, which is the perpendicular bisector of the line segment joining the two poles -1 and 1 of $f(z)$. It appears then that the poles of f have a repelling effect on the zeros of the successive derivatives forcing them, so to speak to align along the y-axis. While we cannot expect things to be so rigid in general, it will turn out that the final set of a meromorphic function with two poles is always the perpendicular bisector of the line segment joining the two poles. This will be an immediate consequence of the main theorem of Pólya on final sets to which we now turn.

2.1 Pólya's Theorem

Theorem 2.1.1 *Suppose that $f(z)$ is a meromorphic function and has at least two distinct poles in $|z - z_0| < R \leq \infty$. Let r be the radius of the largest circle centered at z_0 and containing no poles in its interior.*

(I) If $|z - z_0| = r$ contains only one pole of $f(z)$, then for a sufficiently small positive $\delta, f^{(\ell)}(z) \to \infty$ as $\ell \to \infty$ uniformly in $|z - z_0| \leq \delta$.

(II) If $|z-z_0| = r$ contains at least two distinct poles of $f(z)$, then for every positive δ , the equation $f^{(\ell)}(z) = 0$ has roots in $|z-z_0| \leq \delta$ when ℓ is sufficiently large. [4; 63]

Proof of Part I: Suppose first without loss of generality that $z_0 = 0$ and that ζ_0 is the only pole of f on $|z| = r$. Let $\rho > 0$ be such that $\rho \geq |z| > r$ contains no poles of $f(z)$. One can choose such ρ because of the following: If for every $\rho > 0, \rho \geq |z| > r$ contains poles of $f(z)$ then $\forall \rho > 0, \rho \geq |z| > r$ contains infinitely many poles of $f(z)$. Otherwise, if there exists $\rho > 0$ such that $\rho \geq |z| > r$ contains only finite poles of $f(z)$ and if z^* is the nearest one to the circle $|z| = r$ then $|z^*| \geq |z| > r$ contains no poles of $f(z)$. Hence, since this set of poles is infinite and bounded, it must have a limit point, by Bolzano- Weirstrass theorem. [15;28]. Thus this limit point is an essential singularity of $f(z)$ which is a contradiction.

Let $g_0(z)$ be the principal part of $f(z)$ at $\zeta_0 = r\,e^{i\theta_0}$. Then $\phi(z) = f(z) - g_0(z)$ is analytic in a small neighborhood of ζ_0. Since f and g are analytic in $|z| \leq \rho$ except at ζ_0, then $\phi(z)$ is analytic in $|z| \leq \rho$. Therefore ϕ is continuous there and thus bounded since $|z| \leq \rho$ is compact. Thus there exists a positive integer M such that

$$|\phi(z)| \leq M \qquad \text{for all } |z| \leq \rho.$$

Now let $0 < 3\delta \leq \rho - r$. So if $|z| \leq \delta$ then $|\phi(\zeta)| \leq M$ in $|\zeta - z| \leq r + 2\delta$ (with this in hand $|\zeta|$ will be bounded by ρ;

$$|\zeta| \leq |\zeta - z| + |z| \leq r + 2\delta + \delta \leq \rho$$

and if this is the case then $|\phi(\zeta)| \leq M$). Thus by Cauchy's formula

$$\phi^{\ell}(z) = \frac{\ell!}{2\pi i} \int_{|\zeta - z| = r+2\delta} \frac{\phi(\zeta)}{(\zeta - z)^{\ell+1}} d\zeta.$$

Hence

$$\begin{aligned} |\phi^{(\ell)}(z)| &\leq \frac{\ell!}{2\pi}\int_{|\zeta-z|=r+2\delta}\frac{M}{(r+2\delta)^{\ell+1}}|d\zeta| \\ &= \frac{M\ell!2\pi(r+2\delta)}{2\pi(r+2\delta)^{\ell+1}} = \frac{M\ell!}{(r+2\delta)^{\ell}}\ ; \end{aligned}$$

i.e.

$$|\phi^{(\ell)}(z)| \leq \frac{M\ell!}{(r+2\delta)^{\ell}} \qquad \text{for } |z| \leq \delta.$$

Since $g_0(z)$ is the principal part of $f(z)$ at ζ_0, we can write $g_0(z)$ as follows

$$g_0(z) = \sum_{k=1}^{p}\frac{a_k}{(z-\zeta_0)^k}, \qquad a_p \neq 0,$$

where a_k is a complex constant and then

$$\begin{aligned} g_0^{(\ell)}(z) &= \sum_{k=1}^{p}(-1)^{\ell}\frac{k(k+1)\dots(k+\ell-1)}{(z-\zeta_0)^{k+\ell}}a_k \qquad (2.1) \\ &= (-1)^{\ell}\frac{p(p+1)\dots(p+\ell-1)}{(z-\zeta_0)^{p+\ell}}\left\{a_p+\sum_{k=1}^{p-1}\frac{k(k+1)\dots(k+\ell-1)}{p(p+1)\dots(p+\ell-1)}a_k(z-\zeta_0)^{p-k}\right\}. \end{aligned}$$

Thus

$$|g_0^{(\ell)}(z)| \geq \frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}}\left\{|a_p|-\sum_{k=1}^{p-1}\frac{k(k+1)\dots(k+\ell-1)}{p(p+1)\dots(p+\ell-1)}|a_k|(r+\delta)^{p-k}\right\}$$

since $|z-\zeta_0| \leq |z|+|\zeta_0| \leq \delta + r$.

Suppose $p > 1$ and consider $\frac{k(k+1)\dots(k+\ell-1)}{p(p+1)\dots(p+\ell-1)}$ where $1 \leq k \leq p-1$. We have

$$0 \leq \frac{k(k+1)\dots(k+\ell-1)}{p(p+1)\dots(p+\ell-1)} \leq \frac{(p-1)(p)\dots(p+\ell-2)}{p(p+1)\dots(p+\ell-1)} = \frac{p-1}{p+\ell-1} \qquad (2.2)$$

which tends to zero as ℓ tends to ∞ uniformly in $1 \leq k \leq p-1$. So there exists an integer L such that

$$\sum_{k=1}^{p-1}\frac{k(k+1)\dots(k+\ell-1)}{p(p+1)\dots(p+\ell-1)}|a_k|(r+\delta)^{p-k} \leq \frac{|a_p|}{2} \qquad \text{for all } \ell > L \text{ and all } |z| \leq \delta.$$

Thus

$$|g_0^{(\ell)}(z)| \ \geq \frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}}\left\{|a_p| - \frac{|a_p|}{2}\right\}, \qquad \text{for all } \ell > L \text{ and all } |z| \leq \delta;$$

that is

$$|g_0^{(\ell)}(z)| \ \geq \frac{|a_p|}{2}\frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}}, \qquad \text{for all } \ell > L \text{ and all } |z| \leq \delta.$$

Going back to the original function $f(z)$, it can be written as $f(z) = g_0(z) + \phi(z)$ so that $f^{(\ell)}(z) \ = g_0^{(\ell)}(z) \ + \phi^{(\ell)}(z)$

and

$$\begin{aligned} |f^{(\ell)}(z)\ | \ &\geq \ |g_0^{(\ell)}(z)| \ - |\phi^{(\ell)}(z)| \\ &\geq \ \frac{|a_p|}{2}\frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}} - \frac{M\ell!}{(r+2\delta)^{\ell}}, \end{aligned}$$

uniformly for all $\ell > L$ and $|z| \leq \delta$.

It follows that

$$|f^{(\ell)}(z)\ | \geq \left\{1 - \frac{2M(r+\delta)^{p+\ell}\ell!}{|a_p|p(p+1)\dots(p+\ell-1)\ (r+2\delta)^{\ell}}\right\}\left\{\frac{|a_p|}{2}\frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}}\right\}.$$

But

$$\frac{(r+2\delta)^{\ell}p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}\ \ell!} \geq \left(\frac{r+2\delta}{r+\delta}\right)^{\ell}\frac{1}{(r+\delta)^{p}} \to \infty \text{ as } \ell \to \infty. \qquad (2.3)$$

Thus

$$\frac{2M(r+\delta)^{p+\ell}\ell!}{|a_p|p(p+1)\dots(p+\ell-1)\ (r+2\delta)^{\ell}} \to 0 \text{ as } \ell \to \infty.$$

Therefore

$$|f^{(\ell)}(z)\ | \geq (1+o(1))\left(\frac{|a_p|p(p+1)\dots(p+\ell-1)}{2(r+\delta)^{p+\ell}}\right) \to \infty \text{ as } \ell \to \infty,$$

uniformly in $|z| \leq \delta$; since

$$\frac{p(p+1)\dots(p+\ell-1)}{(r+\delta)^{p+\ell}} \geq \frac{\ell!}{(r+\delta)^{p+\ell}} = \frac{\ell!}{(r+\delta)^{\ell}}\cdot\frac{1}{(r+\delta)^{p}}$$

which tends to ∞ as ℓ tends to ∞ since

$$\frac{(r+\delta)^{\ell}}{\ell!} \to 0 \text{ as } \ell \to \infty$$

(because $\frac{x^k}{k!} \to 0$ as $k \to \infty$ for any x.) This finishes the proof of part I of the theorem.

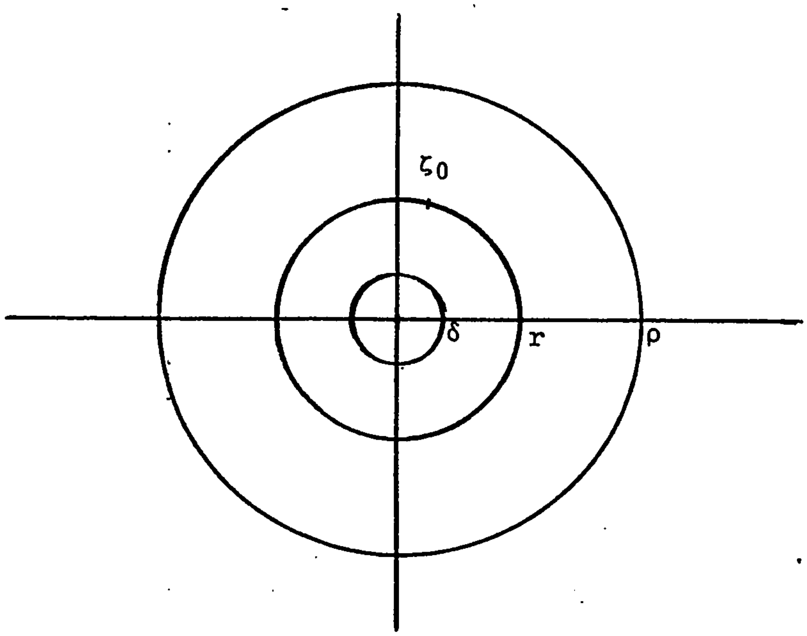

Figure (2.1)

To prove part II of the theorem the following lemma will be needed.

Lemma 2.1.1 *Let $\psi(w)$ be analytic where $|\psi(w)| < 1/2$ for $|w-1| < \eta_0$ where η_0 is a positive real number. Then for large $\ell, w^{\ell} = 1 + \psi(w)$ has at least one root in $|w-1| < \eta_0$. [4;65]*

Proof: To prove this lemma we use the argument principle which simply says that the number of zeros of a function f minus the number of poles of the same function inside a simple closed contour C equals to $\frac{1}{2\pi} var\{arg(f)\}$ along C [14;398]. If f is analytic, we need only show that the variation of $arg(f)$ is positive to conclude that the number of zeros of f is also positive.

Let $\phi(w) = 1 + \psi(w)$ and set $\theta_0 = \pi/\ell, r_1 = e^{-1/\ell}$ and $r_2 = e^{1/\ell}$.

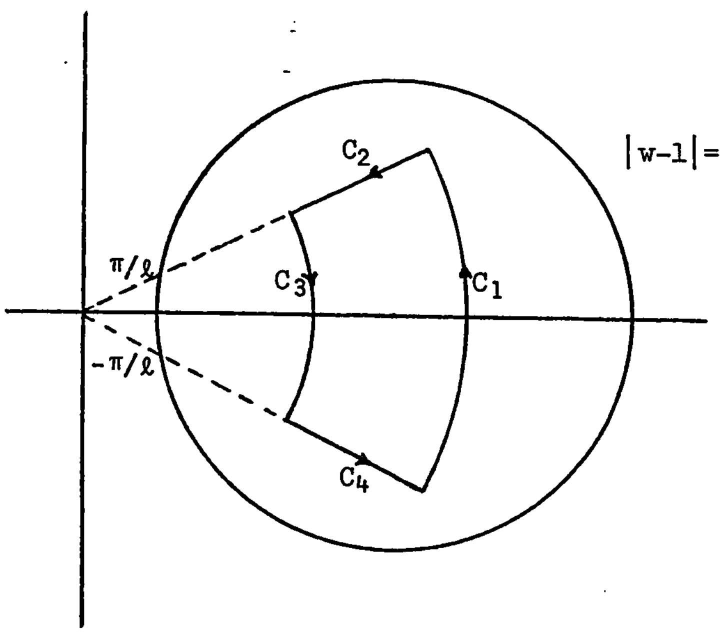

Figure (2.2)

To prove the lemma we shall investigate the variation of the argument of $(\phi(w) - w^\ell)$ along the closed curve $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where

$$C_1 : w = r_2 e^{i\theta} \qquad (-\theta_0 \leq \theta \leq \theta_0),$$

C_2 : the straight line from $r_2 e^{i\theta_0}$ to $r_1 e^{i\theta_0}$,

$$C_3 : w = r_1 e^{-i\theta} \qquad (-\theta_0 \leq \theta \leq \theta_0),$$

C_4 : the straight line from $r_1 e^{-i\theta_0}$ to $r_2 e^{-i\theta_0}$.

Choose ℓ sufficiently large so that C lies in $|w-1| < \eta_0$

Let us consider first the variation of $arg(\phi(w) - w^\ell)$ along the arc C_1.

$$\begin{aligned} arg(\phi(w) - w^\ell) &= arg(-w^\ell(1 - \frac{\phi(w)}{w^\ell})) \\ &= arg(-1) + \ell \cdot arg(w) + arg(1 - \frac{\phi(w)}{w^\ell}). \end{aligned}$$

But $var(arg(-1)) = 0$ along C_1, and

$var(\ell \cdot arg(w)) = \ell(\frac{\pi}{\ell} - \frac{-\pi}{\ell}) = \ell(\frac{2\pi}{\ell}) = 2\pi$ along C_1.

So in order to find the variation of $arg(\phi(w) - w^\ell)$ along C_1 we need to find the variation of $arg(1 - \frac{\phi(w)}{w^\ell})$ along C_1. To do this, let $v = \phi(w)/w^\ell$. Then on C_1 we have

$$|v| = |\phi(w)/w^\ell| \leq \frac{1 + |\psi(w)|}{|w|^\ell} < \frac{1 + 1/2}{r_2^\ell} = \frac{3/2}{(e^{1/\ell})^\ell} = \frac{3}{2e} < 1.$$

So

$$|v| < 1 \text{ on } C_1 \tag{2.4}$$

Now we try to find where $(1 - v)$ is located for all values of v, so that we can find the variation of $arg(1 - v)$.

Suppose that $\pi/2 \leq arg(1 - v) \leq 3\pi/2$, then $|1 - (1 - v)| \geq 1$ or $|v| \geq 1$ which contradicts (2.4). Therefore $arg(1 - v)$ does not lie between $\pi/2$ and $3\pi/2$.

Thus it must lie somewhere between $-\pi/2$ and $\pi/2$.

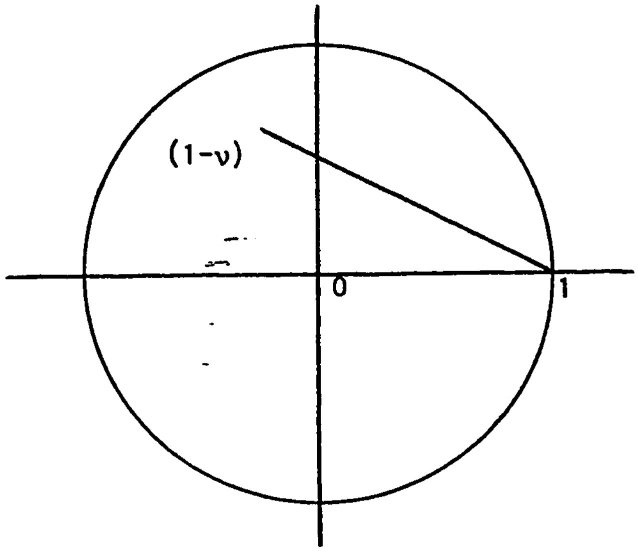

Figure (2.3)

Hence $var(arg(1-v))$ is greater than $-\pi$. Thus

$$var(arg(1-\frac{\phi(w)}{w^\ell})) > -\pi \quad \text{along } C_1.$$

Therefore $var(arg(\phi(w)-w^\ell)) > 0+2\pi-\pi=\pi$ along C_1.

Let us now look at the same argument and try to find its variation on $C_2 \cup C_3 \cup C_4$.

Since by hypothesis $|\psi(w)| < 1/2$ for $|w-1| < \eta_0$ (hence on all C),

$|Re\psi(w)| \leq |\psi(w)| < 1/2$; i.e., $\quad -1/2 < Re\psi(w) < 1/2$.

But on C_2 and C_4, $w^\ell = (re^{\pm i\pi/\ell})^\ell = -r^\ell$, where $e^{-1/\ell} \leq r \leq e^{1/\ell}$. Thus w^ℓ is real and negative on C_2 and C_4.

Hence

$$\begin{aligned} Re(\phi(w)-w^\ell) &= Re(1+\psi(w)-w^\ell) \\ &= 1+Re\psi(w)-Re(w^\ell) \\ &> 1-1/2-Re(w^\ell) \end{aligned}$$

$$= \quad 1 - 1/2 + r^{\ell} > 0 \quad \text{on } C_2 \text{ and } C_4,$$

while on C_3

$$|w^{\ell}| = |e^{-1/\ell}e^{-i\theta}|^{\ell} = |1/e|$$

and so $|w^{\ell}| < 1/2$; consequently $|Re(w^{\ell})| < 1/2$,

which means that $-1/2 < Re(w^{\ell}) < 1/2$.

Hence

$$\begin{aligned} Re(\phi(w) - w^{\ell}) &= 1 + Re\psi(w) - Re(w^{\ell}) \\ &> 1 - 1/2 - 1/2 = 0 \text{ on } C_3. \end{aligned}$$

Therefore on $C_2 \cup C_3 \cup C_4$

$$-\pi/2 < arg(\phi(w) - w^{\ell}) < \pi/2;$$

i.e., $var(arg(\phi(w) - w^{\ell}))$ is greater than $-\pi$ on $C_2 \cup C_3 \cup C_4$.

It follows that $var(arg(\phi(w) - w^{\ell})) > \pi - \pi$ on C and therefore $(\phi(w) - w^{\ell})$ has at least one zero inside C and so inside $|w - 1| < \eta_0$, by the principal argument. Thus for large ℓ, $\quad w^{\ell} = 1 + \psi(w)$ has at least one root in $|w - 1| < \eta_0$.

Proof of Part II: We shall again consider without loss of generality the case where $z_0 = 0$. Also assume that the circle $|z| = r$ has exactly two distinct poles $\zeta_0 = r\,e^{i\theta_0}$ and $\zeta_1 = r\,e^{i\theta_1}$ of $f(z)$ where $0 \leq \theta_0 < \theta_1 < 2\pi$. Let $g_0(z)$ and $g_1(z)$ be the principal parts of f at ζ_0 and ζ_1 respectively.

As before, we suppose that $f(z)$ has no poles in $r < |z| \leq r+3\delta$. So we can write $f(z)$ as

$$f(z) = g_0(z) \; + g_1(z) + \phi(z)$$

where, also as in the proof of the previous part,

$$|\phi^{(\ell)}(z)| \leq \frac{M\ell!}{(r+2\delta)^{\ell}} \qquad \text{uniformly for } |z| \leq \delta.$$

Now , if p and q are the orders of the poles ζ_0 and ζ_1 respectively, then we can show, similar to the previous analysis in part I, that

$$\begin{aligned} g_0^{(\ell)}(z) &= (-1)^{\ell} A_0 \frac{p(p+1)\dots(p+\ell-1)}{(z-\zeta_0)^{p+\ell}} (1+\epsilon_0(z)) \text{ , and} \\ g_1^{(\ell)}(z) &= (-1)^{\ell} A_1 \frac{q(q+1)\dots(q+\ell-1)}{(z-\zeta_1)^{q+\ell}} (1+\epsilon_1(z)) \text{ ,} \end{aligned}$$

where A_0 and A_1 are nonzero complex constants and

$\epsilon_0(z) \to 0$ and $\epsilon_1(z) \to 0$ as $\ell \to \infty$ uniformly for $|z| \leq \delta$.

Let us now consider the equation $f^{(\ell)}(z) = 0$ and try to find out whether or not it has roots in $|z| < \delta$ where ℓ is sufficiently large.

Now

$$f^{(\ell)}(z) = g_0^{(\ell)}(z) \; + g_1^{(\ell)}(z) + \phi^{(\ell)}(z) = 0$$

is equivalent to

$$\begin{aligned} \frac{A_0 p(p+1)\dots(p+\ell-1)\;(1+\epsilon_0(z))}{(z-\zeta_0)^{p+\ell}} &+ \frac{A_1 q(q+1)\dots(q+\ell-1)\;(1+\epsilon_1(z))}{(z-\zeta_1)^{q+\ell}} \\ &+ (-1)^{\ell}\phi^{(\ell)}(z) = 0. \end{aligned} \tag{2.5}$$

If Γ is the gamma function, then since $\Gamma(n) = (n-1)!$ for any positive integer, we can write

$$p(p+1)\ldots(p+\ell-1) \;=\frac{(p+\ell-1)(p+\ell-2)\cdots(p)(p-1)\cdots 2.1}{(p-1)\cdots 2.1}$$

or

$$p(p+1)\ldots(p+\ell-1) \;=\frac{(p+\ell-1)!}{(p-1)!}=\frac{\Gamma(p+\ell)}{\Gamma(p)}.$$

Similarly $q(q+1)\ldots(q+\ell-1) = \dfrac{\Gamma(q+\ell)}{\Gamma(q)}$. Thus we can write (2.5) as follows

$$\frac{A_0\Gamma(p+\ell)\,(1+\epsilon_0(z))}{\Gamma(p)(z-\zeta_0)^{p+\ell}}+\frac{A_1\Gamma(q+\ell))(1+\epsilon_1(z))}{\Gamma(q)(z-\zeta_1)^{q+\ell}}+(-1)^\ell\phi^{(\ell)}(z)=0.$$

Multiplying both sides by $\dfrac{\Gamma(q)(z-\zeta_1)^{q+\ell}}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))}$, we get

$$\frac{A_0\Gamma(p+\ell)\,\Gamma(q)(z-\zeta_1)^{q+\ell}}{\Gamma(p)(z-\zeta_0)^{p+\ell}\,A_1\Gamma(q+\ell)}+\frac{(1+\epsilon_1(z))}{(1+\epsilon_0(z))}+\frac{(-1)^\ell\phi^{(\ell)}(z)\Gamma(q)(z-\zeta_1)^{q+\ell}}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))}=0.$$

Thus

$$-\left(\frac{z-\zeta_1}{z-\zeta_0}\right)^\ell\frac{A_0\Gamma(q)\Gamma(p+\ell)\,(z-\zeta_1)^q}{A_1\Gamma(p)\Gamma(q+\ell)\,(z-\zeta_0)^p}=\frac{(1+\epsilon_1(z))}{(1+\epsilon_0(z))}+\frac{(-1)^\ell\phi^{(\ell)}(z)\Gamma(q)(z-\zeta_1)^{q+\ell}}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))}.$$

Again multiplying both sides by $\dfrac{(z-\zeta_0)^p}{(z-\zeta_1)^q}\dfrac{(-\zeta_1)^q}{(-\zeta_0)^p}$, i.e., by $\dfrac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q}$, we get

$$-\left(\frac{z-\zeta_1}{z-\zeta_0}\right)^\ell\frac{A_0\Gamma(q)\Gamma(p+\ell)\,(-\zeta_1)^q}{A_1\Gamma(p)\Gamma(q+\ell)\,(-\zeta_0)^p}=\frac{(1-\frac{z}{\zeta_0})^p\,(1+\epsilon_1(z))}{(1-\frac{z}{\zeta_1})^q\,(1+\epsilon_0(z))}$$
$$+\frac{(-1)^\ell\phi^{(\ell)}(z)\Gamma(q)(z-\zeta_1)^\ell(-\zeta_1)^q(z-\zeta_0)^p}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))\,(-\zeta_0)^p}. \tag{2.6}$$

Since $|\phi^{(\ell)}(z)|\le\dfrac{M\ell!}{(r+2\delta)^\ell}$ uniformly for $|z|\le\delta$ and

$$|z-\zeta_1|\le|z|+|\zeta_1|\le\delta+r,\qquad\qquad |z-\zeta_0|\le\delta+r;$$

then the modulus of the second term of the right hand side of (2.6) becomes

$$\left|\frac{(-1)^\ell\phi^{(\ell)}(z)\Gamma(q)(z-\zeta_1)^\ell(-\zeta_1)^q(z-\zeta_0)^p}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))\,(-\zeta_0)^p}\right|\le\left|\frac{M\ell!(r+\delta)^\ell\Gamma(q)(-\zeta_1)^q(r+\delta)^p}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))\,(-\zeta_0)^p(r+2\delta)^\ell}\right|.$$

But $\dfrac{\ell!(r+\delta)^\ell}{\Gamma(q+\ell)\,(r+2\delta)^\ell} \to 0 \quad$ as $\ell \to \infty$, $\quad$ by (2.3).

Thus the second term of the right-hand side of (2.6) tends to zero as $\ell \to \infty$, uniformly for $|z| \leq \delta$. Therefore the right-hand side of this equation tends to $\dfrac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q}$ as $\ell \to \infty$.

Now let $w = \dfrac{z-\zeta_1}{z-\zeta_0}C_\ell$, $\quad$ where

$$C_\ell = \left\{\frac{-A_0\Gamma(q)\Gamma(p+\ell)\,(-\zeta_1)^q}{A_1\Gamma(p)\Gamma(q+\ell)\,(-\zeta_0)^p}\right\}^{1/\ell},$$

and so

$$C_\ell(\zeta_1/\zeta_0) = \left\{-\frac{A_0\Gamma(q)\Gamma(p+\ell)\,(-\zeta_1)^q}{A_1\Gamma(p)\Gamma(p+\ell)\,(-\zeta_1)^p}(\zeta_1/\zeta_0)^\ell\right\}^{1/\ell}.$$

where $|arg(C_\ell\frac{\zeta_1}{\zeta_0})| \leq \pi/\ell$.

Note that

$$\left|C_\ell\frac{\zeta_1}{\zeta_0}\right| = |C_\ell| \to 1 \qquad \text{as } \ell \to \infty,$$

and

$$\left|arg(C_\ell\frac{\zeta_1}{\zeta_0})\right| \to 0 \qquad \text{as } \ell \to \infty,$$

thus $C_\ell\frac{\zeta_1}{\zeta_0} \to 1 \quad$ as $\ell \to \infty$.

Consider now the mapping $w(z) = C_\ell\dfrac{z-\zeta_1}{z-\zeta_0}$ which is a linear fractional transformation and thus a one-to-one correspondence mapping with an inverse function equal to

$$w^{-1}(z) = z(w) = \frac{w\zeta_0 - C_\ell\zeta_1}{w - C_\ell}.$$

This inverse function is obviously continuous at $w = 1$, and therefore for every $\gamma > 0$, there exists $\eta > 0$ such that the image of $|w-1| < \eta$, under $z(w)$, is contained in $|z - z(1)| < \gamma$.

Let us now go back to our equation (2.6) and write it in terms of w to get

$$w^{\ell} = \frac{(1-\frac{z}{\zeta_0})^p\,(1+\epsilon_1(z))}{(1-\frac{z}{\zeta_1})^q\,(1+\epsilon_0(z))} + \frac{(-1)^{\ell}\phi^{(\ell)}(z)\,\Gamma(q)(z-\zeta_1)^{\ell}(-\zeta_1)^q(z-\zeta_0)^p}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))\,(-\zeta_0)^p}$$

so

$$w^{\ell} = \frac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q} + o(1) \text{ as } \ell \to \infty \text{ uniformly for } |z| \le \delta.$$

Define ψ by $\psi(w) = w^{\ell} - 1$

and note that

$$\lim_{z\to 0} \frac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q} = 1.$$

So if we choose $\epsilon = \frac{1}{4}$, then there exists $\delta_0 > 0$ such that

$$\left|\frac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q} - 1\right| < \frac{1}{4} \qquad \text{whenever } |z| < \delta_0.$$

Set

$$\frac{(-1)^{\ell}\phi^{(\ell)}(z)\,\Gamma(q)(z-\zeta_1)^{\ell}(-\zeta_1)^q(z-\zeta_0)^p}{A_1\Gamma(q+\ell)\,(1+\epsilon_0(z))\,(-\zeta_0)^p} = F(\ell).$$

So since $F(\ell) = o(1)$ as $\ell \to \infty$ uniformly for $|z| \le \delta$, then for $\epsilon = \frac{1}{4}$ there exists $(0 <)\alpha < \infty$ such that $|F(\ell)| < \frac{1}{4}$ whenever $|\ell| > \alpha$ (or whenever ℓ is sufficiently large.)

Thus

$$|\psi(w)| = |w^{\ell}-1| = \left|\frac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q} + o(1) - 1\right| \le \left|\frac{(1-\frac{z}{\zeta_0})^p}{(1-\frac{z}{\zeta_1})^q} - 1\right| + |o(1)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

whenever $|z| < \min\{\delta_0, \delta\}$ and ℓ is sufficiently large.

So we now have $w^\ell = 1 + \psi(w)$, where $|\psi(w)| < \frac{1}{2}$ if $|z| < \min\{\delta_0, \delta\}$ and δ_0 is sufficiently small positive number and if further ℓ is sufficiently large.

Now, if we choose $\gamma > 0$ so small that $|z - z(1)| < \gamma$ is contained in $|z| < \min\{\delta_0, \delta\}$, then by the continuituy of $z(w)$, there exists $\eta_0 > 0$ such that the image of $|w-1| < \eta_0$ under $z(w)$ is contained in $|z - z(1)| < \gamma$, which is in turn contained in $|z| < \min\{\delta_0, \delta\}$. Thus $w^\ell = 1 + \psi(w)$, where $|\psi(w)| < \frac{1}{2}$ if $|w-1| < \eta_0$, and by the lemma $w^\ell = 1 + \psi(w)$ has at least one root in $|w-1| < \eta_0$. Therefore $f^{(\ell)}(z) = 0$ has at least one root in $|z| < \delta$.

To prove the general case, let $g(z) = f(z_0 + z)$. So if $|z - z_0| = r$ has two poles of $g(z)$ on its circumference and none in its interior then

$$g^{(\ell)}(z - z_0) = f^{(\ell)}(z_0 + z - z_0) = f^{(\ell)}(z) \text{ has a zero in } |z - z_0| < \delta$$

for sufficiently large ℓ.

Finally suppose that $f(z)$ has no poles in $|z - z_0| < r$ but $q(> 2)$ distinct poles on $|z - z_0| = r$, say $z_n = z_0 + re^{i\theta_n}$ where $0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi$.
If we let

$$z' = z_0 + te^{\frac{1}{2}i(\theta_1+\theta_2)}$$

where t is a small positive number, where $r < |z - z_0| < 2t$ has no poles, then $|z - z'| \le |z_1 - z'|$ has exactly two poles of $f(z)$ on its circumference and none in its interior (see Figure (2.4)). So the disk $|z - z'| < t$ contains a zero of $f^{(\ell)}(z)$ for sufficiently large ℓ. Therefore

$$|z - z_0| \le |z - z'| + |z' - z_0| < t + t = 2t$$

contains a zero of $f^{(\ell)}(z)$ for sufficiently large ℓ.

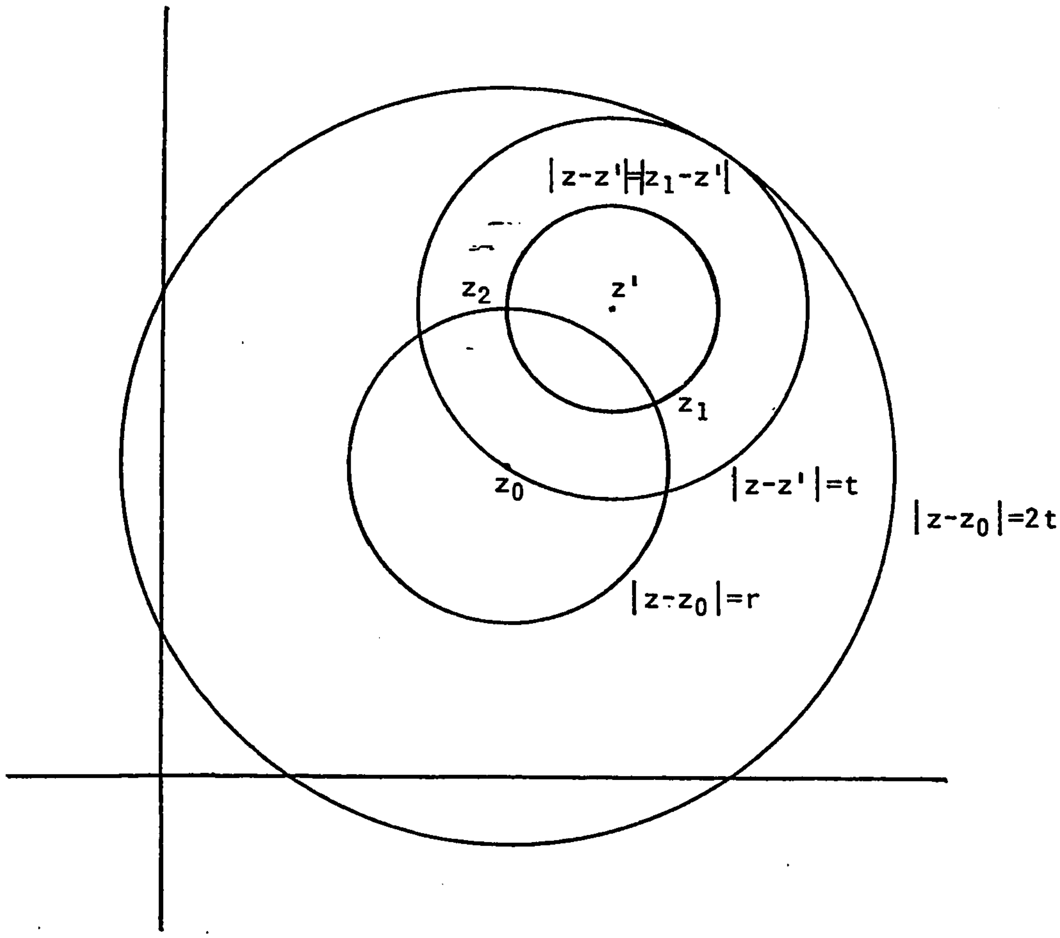

Figure (2.4)

2.2 Theoretical Applications and Examples of Pólya's Theorem

Corollary 2.2.1 *Let $f(z)$ be meromorphic in $|z - z_0| < r$ and has at least two distinct poles. Then $f^{(\ell)}(z)$ has zeros in this disk for sufficiently large ℓ . [4;63]*

Proof: Let $|z - z_0| < r$ be a disk that contains at least two distinct poles of $f(z)$. Let z_1 be the nearest pole of $f(z)$ to z_0.

We shall consider here two cases.

Case I: If there is another pole z_2 such that $|z_2 - z_0| = |z_1 - z_0|$, then $|z - z_0| = |z_1 - z_0|$ contains at least two distinct poles of $f(z)$. Thus, by part II of Pólya's theorem, for every $\delta > 0, f^{(\ell)}(z) = 0$ has roots in $|z - z_0| < \delta$ when ℓ is sufficiently large. In particular, $f^{(\ell)}(z)$ has zeros in the disk $|z - z_0| < r$ for sufficiently large ℓ.

Case II: If there is no pole $\hat{z}$ such that $|\hat{z} - z_0| = |z_1 - z_0|$. Then let z_2 be a pole of $f(z)$ such that the disk $|z - z_0| < |z_2 - z_0|$ contains no poles of $f(z)$ other than z_1, and let $z(t) = (1-t)z_0 + tz_2$ then $z(0) = z_0$ and $z(1) = z_2$.

So $|z(0) - z_1| < |z(0) - z_2|$ and $|z(1) - z_1| > |z(1) - z_2| = 0$. Thus there exists t' such that $0 < t' < 1$ and $|z(t') - z_1| = |z(t') - z_2|$.

Set $z' = z(t')$. So for every pole z^* of $f(z)$ other than z_1 and z_2

$$|z^* - z'| \geq |z^* - z_0| - |z' - z_0| \geq |z_2 - z_0| - |z' - z_0| = |z_2 - z'|,$$

i.e., for any pole z^* of $f(z)$ other than z_1 and z_2

$$|z^* - z'| \geq |z_2 - z'|$$

Therefore $|z - z'| = |z_2 - z'|$ contains at least two poles of $f(z)$ and none in its interior. Thus, by part II of Polya's theorem, for every $\delta > 0, f^{(\ell)}(z)$ has zeros in

$|z - z'| < \delta$ for sufficiently large ℓ. In particular $|z - z_0| < r$, contains such zeros.

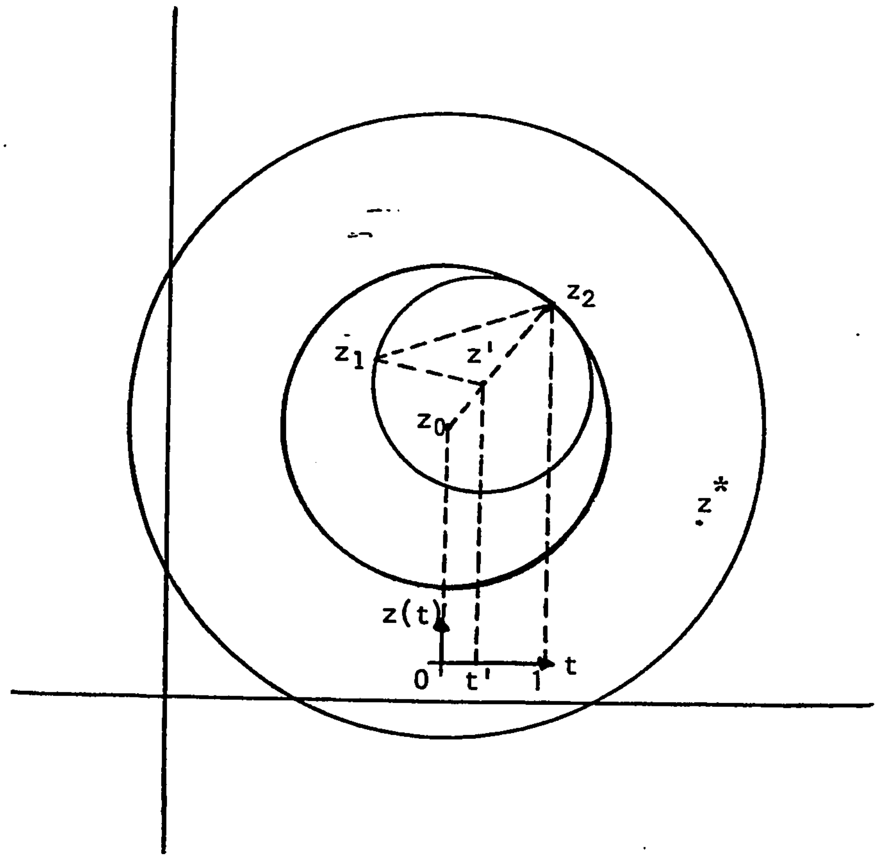

Figure (2.5)

Definition 2.2.1 *Let $f(z)$ be a meromorphic function and let z_0 be a pole of this function. The domain of the pole z_0 consists of all those points of the plane which are nearer to z_0 than to any other pole of $f(z)$. [10;180]*

Remark 2.2.1 *The boundary of the domain of a pole z_0 of a function $f(z)$ consists of those points z satisfying the following:*

(i) For any pole z^* of $f(z)$ other than z_0, $|z - z_0| \leq |z - z^*|$.

(ii) z does not belong to the domain of z_0.

Definition 2.2.2 *The intersection of the boundaries of the domains of any two poles of $f(z)$ is called the common boundary of these domains.*

Note: This common boundary must be contained in the perpendicular bisector of the line segment joining the two poles. [10;180]

Definition 2.2.3 *The union of the common boundaries of the domains of all pairs of poles of $f(z)$ is called the common boundary of $f(z)$.*

Theorem 2.2.1 *Let $f(z)$ be a meromorphic function with $q(\geq 2)$ poles. Then the final set of $f(z)$ is the common boundary of $f(z)$.*

Proof: Let z_0 be any point on the common boundary of $f(z)$, then z_0 must be on the common boundary of the domains of some two poles of $f(z)$, say z_1 and z_2.

Now, let $r = |z_1 - z_0|$, then $|z - z_0| = r$ contains at least two poles, z_1 and z_2 , and none in its interior. To see that there are no poles in the interior of $|z - z_0| = r$, suppose the contrary; i.e., suppose that there are some poles in the interior and let z^* be one of them. Then z_0 will be nearer to z^* than to z_1 or z_2 i.e.,

$$|z_0 - z^*| < |z_0 - z_1| \quad \text{and } |z_0 - z^*| < |z_0 - z_2|.$$

But since z_0 is contained in the common boundary of the domains of z_1 and z_2, then $|z_0 - z_1| \leq |z_0 - z^*|$ and $|z_0 - z_2| \leq |z_0 - z^*|$, by remark (2.2.1), which is a contradiction. Thus $|z - z_0| = r$ has no poles in its interior.

Hence, by part II of Pólya's theorem, for every $\delta > 0, f^{(\ell)}(z) = 0$ has roots in $|z - z_0| < \delta$ for sufficiently large ℓ. This means that z_0 is a limit point of the zeros of

$f^{(\ell)}(z) = 0$ for sufficiently large ℓ. And this proves that every point of the common boundary of $f(z)$ is contained in the final set of $f(z)$.

Conversely, we shall show that no point outside the common boundary of $f(z)$ is contained in the final set of $f(z)$. So, let z_0 be a point outside the common boundary of $f(z)$, then z_0 must be in one of the domains of the poles of $f(z)$. Suppose that z_0 is in the domain of the pole z_1, and let $|z_1 - z_0| = r$, then $|z - z_0| = r$ contains only one pole z_1 of $f(z)$ and none in its interior.

To see that $|z - z_0| = r$ has only one pole z_1, suppose that there is another pole z^* on $|z - z_0| = r$, then $|z_0 - z^*| = |z_0 - z_1|$; i.e., z_0 is on the common boundary of the domains of the two poles $z*$ and z_1 and thus on the common boundary of $f(z)$ which contradicts our choice of z_0.

To see that $|z - z_0| = r$ has no poles in its interior, suppose the contrary; i.e., suppose that there is a pole z^* in its interior, then z_0 is nearer to z^* than to z_1 which means that z_0 is not in the domain of z_1 as was chosen and hence a contradiction.

By part I of Pólya's theorem, for sufficiently small positive $\delta, f^{(\ell)}(z) \to \infty$ as $\ell \to \infty$ uniformly in $|z - z_0| \leq \delta$. Thus z_0 is not a limit point of the zeros of the successive derivatives of $f(z)$. Therefore, no point outside the common boundary of $f(z)$ is contained in the final set of $f(z)$. That is, the final set of $f(z)$ is contained in the common boundary of $f(z)$. This ends the proof of the theorem.

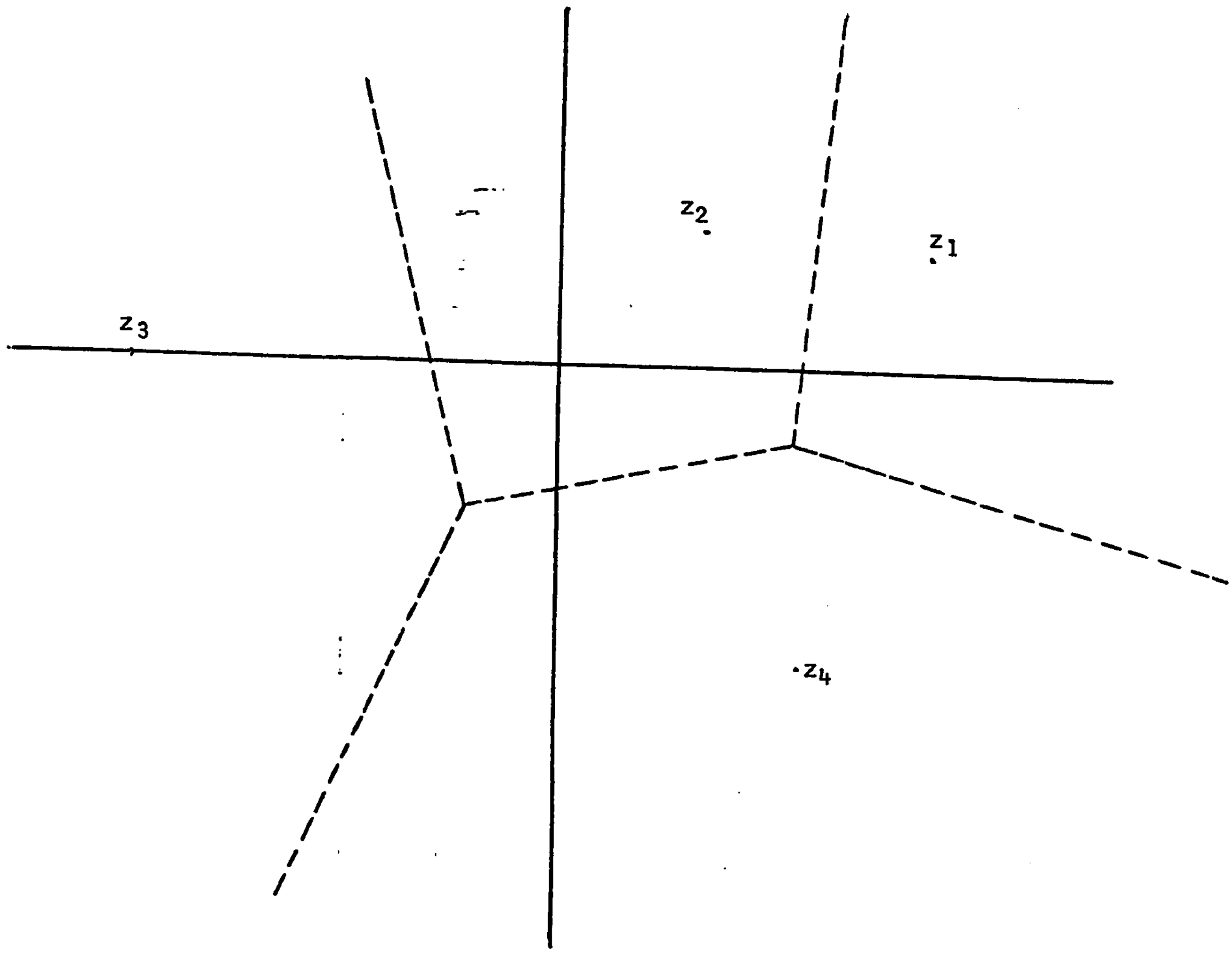

Figure (2.6)

Example 2.2.1 *Let us take the example of Chapter I and try to find its final set using the above theorem.*

The function $f(z) = \frac{2}{z^2 - 1}$ *has only two distinct poles -1 and 1. Hence by the above theorem the final set of this function is the common boundary of* $f(z)$ *which is in this case the perpendicular bisector of the line segment joining the two poles which is the y-axis.*

Example 2.2.2 *Let* $g(z) = \frac{1}{(z^2-1)(z-i-1)}$, *then* $g(z)$ *is a meromorphic function with 1, -1 and (i+1) as its poles. Thus, by the above theorem, the final set of* $g(z)$ *is the common boundary of* $g(z)$ *as shown in the Figure (2.7).*

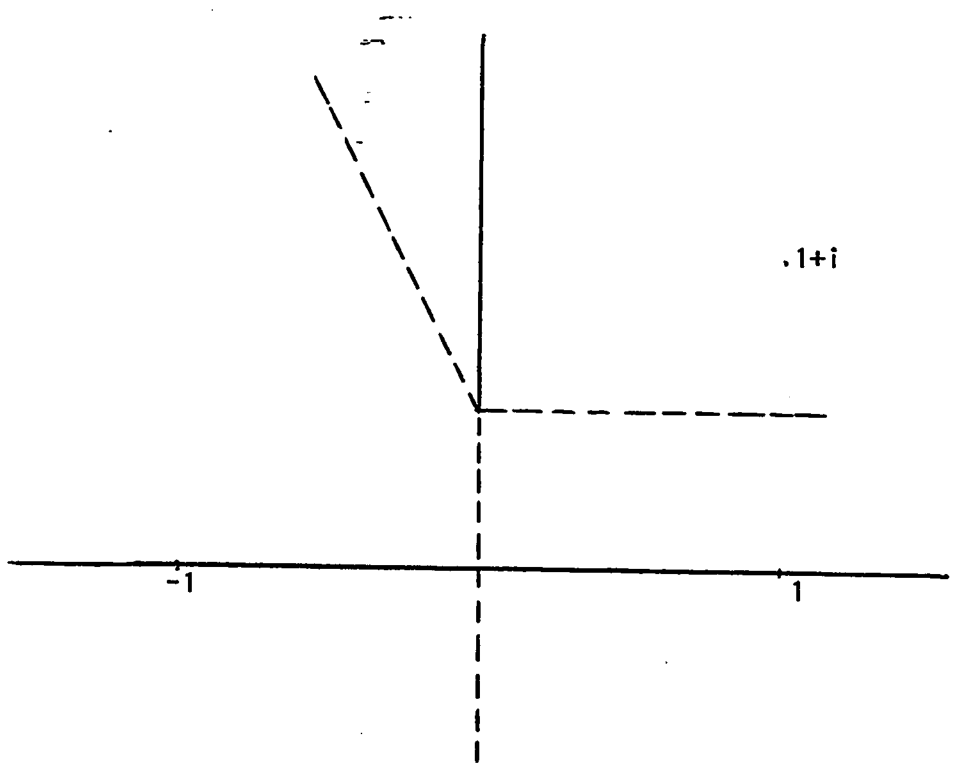

Figure (2.7)

Example 2.2.3 *Let* $h(z) = \frac{1}{\sin z}$, *then* $h(z)$ *is a meromorphic function with poles* $\{n\pi : n \in z\}$. *Thus the final set of* $h(z)$ *is the common boundary of* $h(z)$ *which is the set of perpendicular bisectors of the line segments joining all successive pairs of poles.*

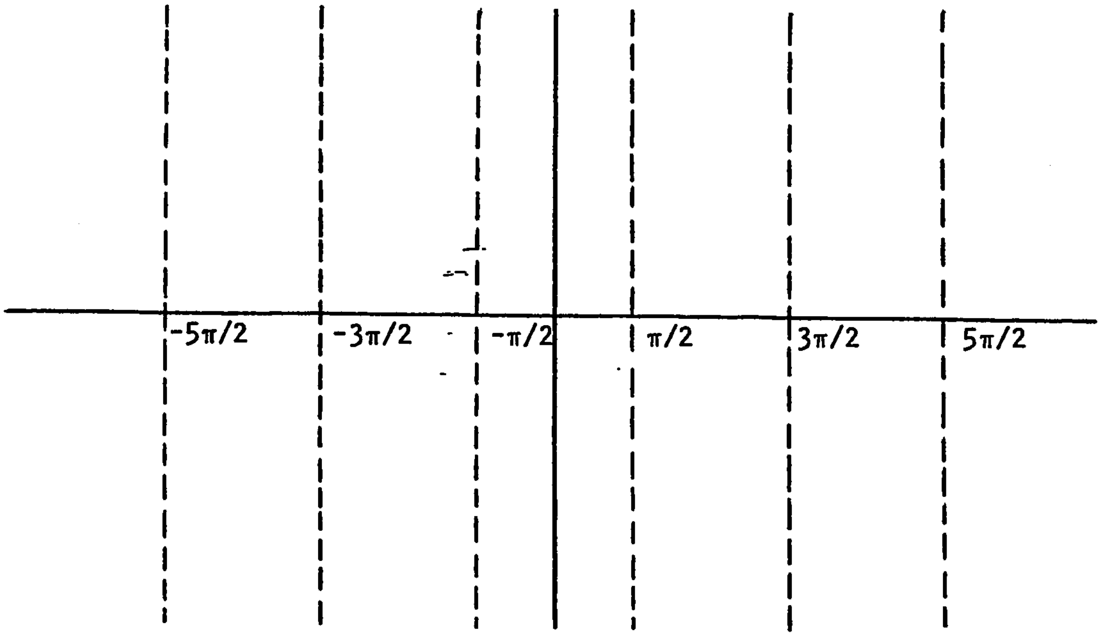

Figure (2.8)

Theorem 2.2.2 *Let $f(z)$ be a meromorphic function with only one pole. Then the final set of $f(z)$ is empty.*

Proof: Let z_0 be the only pole of $f(z)$ and let z^* be any point in the complex plane. Let $r = |z^* - z_0|$, then $|z - z^*| = r$ contains only one pole z_0 of $f(z)$ which is the only one of $f(z)$. Hence, by part I of Polya's theorem, for sufficiently small positive δ, $f^{(\ell)}(z) \rightarrow \infty$ as $\ell \rightarrow \infty$ uniformly in $|z - z^*| \leq \delta$. Therefore, every point in the complex plane is contained in some neighborhood that does not contain zeros of infinitely many among the successive derivates $f'(z), f''(z), \cdots$ of $f(z)$. Hence no point of the complex plane is a limit point of the set of zeros of the successive derivates of $f(z)$. Thus the final set of $f(z)$ is the empty set.

Example 2.2.4 *Let $f(z) = \frac{e^z}{z}$. Then the final set of $f(z)$ is the empty set.*

We note that if f and all its derivatives have no zeros, then f must take a simple form as the following theorem shows.

Theorem 2.2.3 *Let $f(z)$ be a meromorphic function where f and all its successive derivatives $f'(z), f''(z), f'''(z), \cdots$ have no zeros. Then $f(z)$ will either be of the form $f(z) = e^{az+b}$ or $f(z) = (az \mp b)^{-n}$ where $a \neq 0, a$ and b are constants and n is a positive integer. [4; 66]*

Chapter 3

The Zeros of Some Classical Polynomials

We have seen in Chapter 2 that the final set of a meromorphic function is completely determined by the poles of the function. For general entire function, no characterization of the final set is yet known. We shall therefore consider, instead, the question of the location of zeros of the successive derivatives of an entire function and also of some meromorphic functions. This question is of great interest and difficulty. But, in certain cases, it turns out to be intimately connected to the question of the location of zeros of certain classical polynomials.

3.1 The Zeros of the Polynomial $P_n(t) = \sum_{k=1}^{n} k! S_k^n t^k$

The Stirling numbers of the second kind will be used frequently throughout this section and the following section. So a quick review of this notion will be given in advance.

Definition 3.1.1 *Let S be a set of n elements and let P be a partition into k classes formed by k disjoint non-empty subsets of S whose union is the whole set S. Then S_k^n is the number of different partitions into k classes of S. This number S_k^n is called* *the Stirling number of the second kind. [12;42]*

Remark 3.1.1 *(i)* $S_1^1 = 1$ *and* $S_n^n = 1$.

(ii) $S_k^{n+1} = S_{k-1}^n + kS_k^n$ *, n and k are positive integers.*

(iii) $S_k^n = 0$ *if* $n < k$ *or* $k < 1$.

(iv) The following table gives some values of S_k^n *for some integer n and some integer* k :

S_k^n	1	2	3	4	5	$\cdots$
1	1	1	1	1	1	$\cdots$
2	0	1	3	7	15	$\cdots$
3	0	0	1	6	25	$\cdots$
4	0	0	0	1	10	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Remark 3.1.2 *Let* $a_k^n = k!S_k^n$

Then

$$\begin{aligned} a_k^{n+1} &= k!S_k^{n+1} \\ &= k!(S_{k-1}^n + kS_k^n) \\ &= k((k-1)!S_{k-1}^n + k!S_k^n) \\ &= k(a_{k-1}^n + a_k^n). \end{aligned}$$

Proposition 3.1.1 *Let* $P_n(t) = \sum_{k=1}^{n} k!S_k^n t^k$ *, where* S_k^n *is the Stirling number of the second kind. Then*

$$P_{n+1}(t) = t \cdot \frac{d}{dt}[(t+1)P_n(t)].$$

Proof: By Remark (3.1.2) we can write P_n as follows:

$$P_n(t) = \sum_{k=1}^{n} a_k^n t^k.$$

So

$$
\begin{aligned}
P_{n+1}(t) &= \sum_{k=1}^{n+1} a_k^{n+1}\, t^k \\
&= \sum_{k=1}^{n+1} k(a_{k-1}^n + a_k^n)t^k \qquad \text{by Remark (3.1.2)} \\
&= \sum_{k=0}^{n} (k+1)(a_k^n + a_{k+1}^n)t^{k+1} \\
&= t\left[\sum_{k=0}^{n} a_k^n\, t^k + \sum_{k=0}^{n} k a_k^n\, t^k + \sum_{k=0}^{n} (k+1) a_{k+1}^n t^k\right] \\
&= t\left[\sum_{k=1}^{n} a_k^n t^k + \sum_{k=1}^{n} k a_k^n\, t^k + \sum_{k=1}^{n+1} k a_k^n\, t^{k-1}\right] \qquad \text{since } a_0^n = 0 \\
&= t\left[\sum_{k=1}^{n} a_k^n t^k + (t+1)\sum_{k=1}^{n} k a_k^n\, t^{k-1}\right] \qquad \text{since } a_{n+1}^n = 0 \\
&= t\left[P_n(t) + (t+1)\frac{d}{dt}P_n(t)\right] \\
&= t\cdot\frac{d}{dt}\left[(t+1)P_n(t)\right].
\end{aligned}
$$

Hence $P_{n+1}(t) = t\cdot\frac{d}{dt}[(t+1)P_n(t)]$ as desired.

Theorem 3.1.1 *Let $P_n(t) = \sum_{k=1}^{n} k!S_k^n t^k$. Then the zeros of $P_n(t)$ are real, simple and, except for the zero at the origin, contained in the interval $(-1, 0)$.*

Proof: We will use mathematical induction to prove this theorem.

For $n = 1$, $\qquad P_1(t) = t$

thus $P_1(t) = 0$ if and only if $t = 0$.

Hence the zero of $P_1(t)$ is real and simple.

For $n = 2$, $\qquad P_2(t) = \sum_{k=1}^{2} k! S_k^n \, t^k = t + 2t^2 = t(1 + 2t)$.

So $P_2(t) = 0$ if and only if $t = 0 or - 1/2$ and again the zeros of $P_2(t)$ are real, simple and except for $t = 0$, contained in the interval $(-1, 0)$.

We now assume that the hypothesis is true for $P_n(t)$, and we need only show that it is true for $P_{n+1}(t)$.

By hypothesis, $(t + 1)P_n(t)$ has $(n + 1)$ real simple zeros including $t = -1$ and $t = 0$ and, except for these two zeros, the zeros are contained in the interval $(-1, 0)$.

To complete the proof, we shall use Rolle's theorem which says that "if f is a continuous function on $[a, b]$ which is differentiable on (a, b) and satisfies $f(a) = f(b)$. Then there exists at least one x in (a, b) such that $f'(x) = 0$. " [13;162]

Using this theorem, $\frac{d}{dt}[(t + 1)P_n(t)]$ must have at least n different zeros in the interval $(-1, 0)$. Therefore $P_{n+1}(t) = t \cdot \frac{d}{dt}[(t + 1)P_n(t)]$ has at least $(n+1)$ different zeros, but since it is a polynomial of degree $(n + 1)$, it must have exactly $(n + 1)$ different zeros. Hence all the zeros of P_{n+1} are real, simple and except for $t = 0$, contained in the interval $(-1, 0)$.

3.2 The Zeros of the Polynomial $Q_n(t) = \sum_{k=1}^{n} S_k^n t^k$

Proposition 3.2.1 *Let* $Q_n(t) = \sum_{k=1}^{n} S_k^n t^k$ *, where* S_k^n *is the Stirling number of the second kind. Then*

$$Q_{n+1}(t) = te^{-t} \frac{d}{dt}[e^t Q_n(t)] \; .$$

Proof:

$$Q_{n+1}(t) \quad = \quad \sum_{k=1}^{n+1} S_k^{n+1} t^k$$

$$
\begin{aligned}
&= \sum_{k=1}^{n+1} S_{k-1}^n t^k + \sum_{k=1}^{n+1} kS_k^n\, t^k \qquad \text{by Remark (3.1.1 (ii))} \\
&= \sum_{k=0}^{n+1} S_k^n\, t^{k+1} + \sum_{k=1}^{n} kS_k^n\, t^k, \qquad \text{since } S_{n+1}^n = 0 \\
&= t\sum_{k=1}^{n} S_k^n\, t^k + t\sum_{k=1}^{n} kS_k^n\, t^{k-1}, \qquad \text{since } S_0^n = 0
\end{aligned}
$$

Hence $Q_{n+1}(t) = t[Q_n(t) + \frac{d}{dt}Q_n(t)\,]$.

We now multiply both sides by e^t to get

$$
\begin{aligned}
e^tQ_{n+1}(t) &= t[e^tQ_n(t) + e^t\frac{d}{dt}Q_n(t)\,] \\
&= t\frac{d}{dt}[e^tQ_n(t)\,].
\end{aligned}
$$

Thus $Q_{n+1}(t) = te^{-t}\frac{d}{dt}[e^tQ_n(t)]$ as required.

Theorem 3.2.1 *Let* $Q_n(t) = \sum_{k=1}^{n} S_k^n t^k$ *. Then the zeros of* $Q_n(t)$ *are real, simple and nonpositive.*

Proof: As in the proof of theorem (3.1.5), mathematical induction will be used to prove this result.

For $n = 1$, $\quad P_1(t) = t$.

Thus $P_1(t) = 0$ if and only if $t = 0$.

Hence the zeros of $P_1(t)$ are real, simple and nonpositive.

For $n = 2$, $\quad P_2(t) = t + t^2$.

Thus $P_2(t) = 0$ if and only if $t = 0$ or $t = -1$.

Hence the zeros of $P_2(t)$ are real, simple and nonpositive.

We now assume that $Q_n(t)$ has n simple, real and nonpositive zeros, say $t_1, t_2, \cdots, t_n$ where $0 = t_1 > t_2 > \cdots > t_n$, and show that the hypotheis is also true for Q_{n+1}.

The zeros of $Q_{n+1}(t)$ are the same as those of $\frac{d}{dt}[e^tQ_n(t)]$ together with $t = 0$, since $Q_{n+1}(t) = te^{-t}\frac{d}{dt}[e^tQ_n(t)]$. Thus, we need only find the zeros of $\frac{d}{dt}[e^tQ_n(t)]$. The polynomial $Q_n(t)$ and so the function $e^tQ_n(t)$ has n real, simple and non-positive zeros, by hypothesis. Thus, applying Rolle's theorem in the intervals $(t_k, t_{k-1}), k = 2, \cdots, n$; $\frac{d}{dt}[e^tQ_n(t)]$ will have at least $(n-1)$ distinct zeros, say $s_1, s_2, \cdots, s_{n-1}$ where $s_1 > s_2 > \cdots > s_{n-1}$. Thus $te^{-t}\frac{d}{dt}[e^tQ_n(t)]$ must have at least n distinct zeros.

The polynomial $Q_{n+1}(t) = te^{-t}\frac{d}{dt}[e^tQ_n(t)]$ must have $(n+1)$ zeros, because it is of degree $(n+1)$. Thus there must be one more zero, say s_n of $\frac{d}{dt}[e^tQ_n(t)]$. Where could this missing zero be located?

To answer this particular question, let us study the behavior of $e^tQ_n(t)$ as $t \to -\infty$.

$$\lim_{t\to-\infty} e^tQ_n(t) = \lim_{s\to\infty} \frac{Q_n(-s)}{e^s} = 0$$

Hence, if $\frac{d}{dt}[e^tQ_n(t)] > 0$ for all $t < t_n$ then $e^tQ_n(t)$ will be an increasing function for $t < t_n$. So

$$0 = \lim_{t\to-\infty} e^tQ_n(t) \leq e^tQ_n(t) \leq e^{t_n}Q_n(t_n) = 0, t < t_n;$$

i.e., $e^tQ_n(t) = 0$ for all $t < t_n$ which is a contradiction.

Similarly, if $\frac{d}{dt}[e^tQ_n(t)] < 0$ for all $t < t_n$, then $e^tQ_n(t)$ will be a decreasing function for $t < t_n$. So

$$0 = \lim_{t\to-\infty} e^tQ_n(t) \geq e^tQ_n(t) \geq e^{t_n}Q_n(t_n) = 0, t < t_n;$$

i.e., $e^tQ_n(t) = 0$ for all $t < t_n$ which is a contradiction. Therefore, there must be at least one point $t < t_n$ where $\frac{d}{dt}[e^tQ_n(t)] = 0$.

This zero, say s_n, is the only point less than t_n that satisfies $\frac{d}{dt}[e^tQ_n(t)] = 0$, otherwise $Q_{n+1}(t)$ will have more than $(n+1)$ zeros, which is not true. Thus Q_{n+1} has $(n+1)$ real, simple and nonpositive zeros. So the theorem is proved.

3.3 The Zeros of Hermite Polynomials

Definition 3.3.1 *A polynomial $H_n(z)$ is called a Hermite polynomial if it satisfies the conditions*

$$\int_{-\infty}^{\infty} e^{-z^2} H_m(z)H_n(z)dz = \begin{cases} 0, & m \neq n \\ 2^n n!\sqrt{\pi}, & m = n \end{cases} \quad [16; 105]$$

Properties of Hermite Polynomials:

1. $H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$.
2. $H_n'(z) = 2nH_{n-1}(z)$.
3. $H_n(z) = 2zH_{n-1}(z) - H_{n-1}'(z)$.
4. $H_n(z)$ satisfies the Hermite differential equation $y'' - 2zy' + 2ny = 0$.
5. The coefficients of $H_n(z)$ are real.

Lemma 3.3.1 *Let $H_n(t)$ be an orthogonal polynomial of degree n. Then $H_n(t)$ is orthogonal to any polynomial of degree less than n.*

Proof: We shall show first that any polynomial of degree $\leq n$ can be represented as a linear combination of any finite sequence of polynomials of degrees 0 to n.

Let $P_0, P_1, \cdots, P_n$ be polynomials such that

$$P_0(x) = a_0^0, \qquad a_0^0 \neq 0$$

$$
\begin{aligned}
P_1(x) &= a_0^1 + a_1^1 x, \qquad a_1^1 \neq 0 \\
&\vdots \\
P_n(x) &= a_0^n + a_1^n x + \cdots + a_n^n x^n, \qquad a_n^n
\end{aligned}
$$

and let $Q_m(x)$ be a polynomial of degree $m \leq n$ such that
$Q_m(x) = b_0 + b_1 x + \cdots + b_m x^m$.

We shall find $c_0, c_1, \cdots, c_n$ such that

$$Q_m(x) = c_0 P_0(x) + c_1 P_1(x) + \cdots + c_n P_n.$$

Thus

$$
\begin{aligned}
b_0 + b_1 x + \cdots + b_m x^m &= (c_0 a_0^0 + c_1 a_0^1 + \cdots + c_n a_0^n) + (c_1 a_1^1 + \cdots + c_n a_1^n) x \\
&+ \cdots + (c_m a_m^m + \cdots + c_n a_m^n) x^m + \cdots + (c_n a_n^n) x^n.
\end{aligned}
$$

Thus $c_k a_k^k + \cdots + c_n a_k^n = 0$ for all $k > m$, which implies that $c_k = 0$ for all $k > m$.
Thus $b_m = c_m a_m^m$, so $c_m = \frac{b_m}{a_m^m}$, and

$$b_{m-1} = c_{m-1} a_{m-1}^{m-1} + c_m a_{m-1}^m, \text{ so } c_{m-1} = \frac{b_{m-1} - c_m a_{m-1}^m}{a_{m-1}^{m-1}}.$$

Hence, inductively

$$c_k = \frac{b_k - c_{k+1} a_k^{k+1} - c_{k+2} a_k^{k+2} - \cdots - c_m a_k^m}{a_k^k}, \qquad k \leq m.$$

Therefore, any polynomial can be written as a linear combination of any finite sequence of polynomials.

Thus we can write any polynomial $Q_n(t)$ of degree $(n-1)$ as a linear combination of $H_0(t), H_1(t), \cdots, H_{n-1}(t)$ as follows:

$$Q_n(t) = c_0 H_0 + \cdots + c_{n-1} H_{n-1}(t),$$

for some constants $c_i, i = 1, 2, \cdots, n-1$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} H_n(t)Q_n(t)\, e^{-t^2} dt &= c_0 \int_{-\infty}^{\infty} H_0(t)H_n(t)e^{-t^2} dt + \cdots \\ &+ c_{n-1} \int_{-\infty}^{\infty} H_{n-1}(t)H_n(t)e^{-t^2} \\ &= 0.\end{aligned}$$

Thus $H_n(t)$ is orthogonal to any polynomial of degree less than n.

Theorem 3.3.1 *The zeros of a Hermite polynomial $H_n(t)$ are real and simple. [16; 45]*

Proof: Let z_0 be a zero of $H_n(t)$. We shall show that z_0 is actually a real zero. Since the coefficients of $H_n(t)$ are real, $\bar{z}_0$ is also a zero of $H_n(t)$. Then $\frac{H_n(t)}{t-\bar{z}_0}$ is a polynomial of degree $(n-1)$. And since $H_n(t)$ is orthogonal to any polynomial of degree less than n, by the above lemma , then

$$\int_{-\infty}^{\infty} H_n(t)\frac{H_n(t)}{t-\bar{z}_0}e^{-t^2} dt = 0.$$

Thus

$$\int_{-\infty}^{\infty} (t-z_0)\left|\frac{H_n(t)}{t-z_0}\right|^2 e^{-t^2} dt = 0.$$

or

$$\int_{-\infty}^{\infty} t\left|\frac{H_n(t)}{t-z_0}\right|^2 e^{-t^2} dt = z_0 \int_{-\infty}^{\infty} \left|\frac{H_n(t)}{t-z_0}\right|^2 e^{-t^2} dt.$$

Hence z_0 must be real, because the integral of the right hand side of this equation is positive.

Now suppose that z_0 is a multiple zero, then, by using the orthogonality of H_n to any polynomial of degree less than n,

$$\int_{-\infty}^{\infty} H_n(t)\frac{H_n(t)}{(t-z_0)^2}e^{-t^2} dt = 0.$$

Thus

$$\int_{-\infty}^{\infty} \left(\frac{H_n(t)}{t - z_0}\right)^2 e^{-t^2} dt = 0,$$

which is a contradiction, since the integral of the left hand side of this equation is positive.

Chapter 4

The Zero Sets of Successive Derivatives of Some Functions

In this chapter we will find the zero sets of the successive derivatives of some meromorphic and entire functions using the results of the previous chapter. This will be done after we find the relations between the zeros of the functions and the zeros of the corresponding polynomials.

4.1 The Zeros of the Successive Derivatives of $f(z) = (a - be^z)^{-1}$

Proposition 4.1.1 *Let F be a differentiable function having n derivatives. Then the nth derivative of $F(e^z)$ is of the form*

$$\left(\frac{d}{dz}\right)^n F(e^z) = \sum_{k=1}^{n} S_k^n F^{(k)}(e^z)e^{kz} \tag{4.1}$$

where S_k^n is the Stirling number of the second kind.

Proof: In order to prove the proposition, we will use mathematical induction on n.

For $n = 1$, $\quad \frac{d}{dz}F(e^z) = F'(e^z)e^z = S_1^1 F'(e^z)e^z.$

Thus the result is true for $(n = 1)$.

Let us now assume that formula (4.1) is true for $(n-1)$, i.e.,

$$(\frac{d}{dz})^{n-1}F(e^z) = \sum_{k=1}^{n-1} S_k^{n-1}F^{(k)}(e^z)e^{kz} \tag{4.2}$$

and show that it is also true for n.

If we differentiate both sides of equation (4.2) we get

$$\begin{aligned}
(\frac{d}{dz})^n F(e^z) &= \sum_{k=1}^{n-1} S_k^{n-1}F^{(k+1)}(e^z)e^{(k+1)z} + \sum_{k=1}^{n-1} kS_k^{n-1}F^{(k)}(e^z)e^{kz} \\
&= \sum_{k=2}^{n} S_{k-1}^{n-1}F^{(k)}(e^z)e^{kz} + \sum_{k=1}^{n} kS_k^{n-1}F^{(k)}(e^z)e^{kz}, \text{ since } S_n^{n-1} = 0 \\
&= \sum_{k=1}^{n} \left[S_{k-1}^{n-1}F^{(k)}(e^z)e^{kz} + kS_k^{n-1}F^{(k)}(e^z)e^{kz}\right], \text{ since } S_0^{n-1} = 0 \\
&= \sum_{k=1}^{n}(S_{k-1}^{n-1} + kS_k^{n-1})F^{(k)}(e^z)e^{kz}.
\end{aligned}$$

Thus , remark (3.1.1(ii)) implies that

$$(\frac{d}{dz})^n F(e^z) = \sum_{k=1}^{n} S_k^n F^{(k)}(e^z)e^{kz}$$

as required.

Lemma 4.1.1 *Let $g(z) = (a - be^z)^{-1}$ where a and b are non-zero complex numbers. Then*

$$(\frac{d}{dz})^n g(z) = (a - be^z)^{-1}\ P_n(t)$$

where $P_n(t) = \sum_{k=1}^{n} k! S_k^n t^k$ *and* $t = \dfrac{be^z}{a - be^z}$.

Proof: Let $G(u) = (a - bu)^{-1}$. Then the nth derivative of G is

$$G^{(n)}(u) = n!b^n(a - bu)^{-(n+1)}.$$

We can also see that $G(e^z) = g(z)$, thus

$$(\frac{d}{dz})^n g(z) = (\frac{d}{dz})^n G(e^z).$$

Hence, using the above proposition,

$$\begin{aligned}
(\frac{d}{dz})^n g(z) &= (\frac{d}{dz})^n G(e^z) \\
&= \sum_{k=1}^{n} S_k^n \; G^{(k)}(e^z) e^{kz} \\
&= \sum_{k=1}^{n} k! S_k^n \; b^k (a - be^z)^{-(k+1)} e^{kz} \\
&= (a - be^z)^{-1} \sum_{k=0}^{n} k! S_k^n \left(\frac{be^z}{a - be^z}\right)^k \\
&= (a - be^z)^{-1} \sum_{k=1}^{n} k! S_k^n t^k
\end{aligned}$$

where $t = \frac{be^z}{a - be^z}$. Therefore $(\frac{d}{dz})^n (a - be^z)^{-1} = (a - be^z)^{-1} \, P_n(t)$ as required.

This lemma also tells us that the zeros of the nth derivative of $(a - be^z)^{-1}$ are the same as those of $P_n(t) = \sum_{k=1}^{n} k! S_k^n t^k$ where $t = \frac{be^z}{a - be^z}$. Since the zeros of $P_n(t)$ are real, simple and, except for the zero at the origin, contained in the interval $(-1, 0)$, by theorem (3.1.1) , then we need only find the conditions on z when $t = \frac{be^z}{a - be^z}$ is a zero of $P_n(t)$.

Lemma 4.1.2 *If $t = \frac{be^z}{a - be^z}$ where a and b be non-zero complex numbers, then $z \in \{z = x + iy : y = (2k-1)\pi - arg(\bar{a}b), k = 0, \pm 1, \pm 2, \cdots\}$ if and only if t is real and satisfies the inequality $-1 < t < 0$.*

Proof: We know that t is real $\Leftrightarrow t = \bar{t}$

$$\begin{aligned}
&\Leftrightarrow \frac{be^z}{(a - be^z)} = \frac{\bar{b}e^{\bar{z}}}{(\bar{a} - \bar{b}e^{\bar{z}})} \\
&\Leftrightarrow \bar{a}be^z = a\bar{b}e^{\bar{z}} \\
&\Leftrightarrow \frac{\bar{a}b}{a\bar{b}} e^{2yi} = 1.
\end{aligned}$$

Set $w = \bar{a}b = e^{\phi + i\theta}$. Then $|w| = e^{\phi}$ and $\arg(w) = \theta$.

Thus $w/\bar{w}e^{2yi} = 1$

$$\begin{aligned}
&\Leftrightarrow \quad e^{2\theta i}e^{2yi} = 1 \\
&\Leftrightarrow \quad e^{2(\theta+y)i} = 1 \\
&\Leftrightarrow \quad 2(\theta + y) = 2k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots \\
&\Leftrightarrow \quad y = k\pi - arg(w). \qquad\qquad (4.3)
\end{aligned}$$

Therefore t is real $\Leftrightarrow z \in \{z = x + iy : y = k\pi - arg(w), k = 0, \pm 1, \pm 2, \cdots\}$.

We can also notice that through the above argument we have found that t is real if and only if $\bar{a}be^{z} = a\bar{b}e^{\bar{z}}$. That is $\bar{a}be^{z}$ is real, i.e., we^{z} is real.

Let us now take a look at t when it is contained in the interval $(-1, 0)$. Set $u = we^{z}$. Then

$$t = \frac{be^{z}}{a - be^{z}} = \frac{\bar{a}be^{z}}{|a|^2 - \bar{a}be^{z}} = \frac{we^{z}}{|a|^2 - we^{z}} = \frac{u}{|a|^2 - u}.$$

$$\Rightarrow u = \frac{|a|^2 t}{1 + t}.$$

Thus if $-1 < t < 0$, then $u = \frac{|a|^2 t}{1+t} < 0$.

Therefore t is real and satisfies the inequality $-1 < t < 0$ if and only if u is real and negative.

Now, since u is negative, $we^{z} < 0$ and so $e^{(\phi+x)+(\theta+y)i} < 0$. Thus $e^{(\theta+y)i} < 0$. Equation (4.3) says that $y = k\pi - \theta$, hence $e^{k\pi i} < 0, k = 0, \pm 1, \pm 2, \cdots$. Therefore $e^{(\theta+y)i}$ must equal to -1 and so $\theta + y = (2k-1)\pi, k = 0, \pm 1, \pm 2, \cdots$. Hence t is real and contained in the interval $(-1,0)$ if and only if

$$z \in \{z = x + iy : y = (2k-1)\pi - arg(\bar{a}b), k = 0, \pm 1, \pm 2, \cdots\} .$$

Theorem 4.1.1 *All the zeros of all the successive derivatives of* $f(z) = (a - be^z)^{-1}$ *are contained in the set* $\{z = x + iy : y = (2k-1)\pi - arg(\bar{a}b), k = 0, \pm 1, \pm 2, \cdots\}$.

Proof: $(\frac{d}{dz})^n (a - be^z)^{-1} = 0$

$\Leftrightarrow$ $(a - be^z)^{-1} P_n(t) = 0$ where $t = \dfrac{be^z}{a - be^z}$, by proposition (4.1.1)

$\Leftrightarrow$ $P_n(t) = 0,$

$\Leftrightarrow$ t is real, simple and $-1 < t < 0$, by theorem (3.1.1)

$\Leftrightarrow$ $z \in \{z = x + iy : y = (2k-1)\pi - arg(\bar{a}b), k = 0, \pm 1, \pm 2, \cdots\}$, by lemma (4.1.2).

Therefore the zeros of all the derivatives of $g(z) = (a - be^z)^{-1}$ are contained in the set $\{z = x + iy : y = (2k-1)\pi - arg(\bar{a}b), k = 0, \pm 1, \pm 2, \cdots\}$.

The following theorem is a nice application of the above theorem suggested by George Pólya [11; 455] and I find it interesting to prove it and include it here.

Theorem 4.1.2 *Let* $f(w) = \tan w$. *If the nth derivative of* $f(w)$ *vanishes at a point* w, *then* w *is a simple zero of that derivative and the real part of* w *is a multiple integer of* π.

Proof: Consider the function $f(\frac{\pi}{2} - z)$, then

$$\begin{aligned} f(\frac{\pi}{2} - z) &= \tan(\frac{\pi}{2} - z) \\ &= \cot z \\ &= \frac{(e^{iz} + e^{-iz})/2}{(e^{iz} - e^{-iz})/2i} \end{aligned}$$

$$
\begin{aligned}
&= i\left(\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right)\\
&= i\left(\frac{e^{2iz}+1}{e^{2iz}-1}\right)\\
&= \frac{i(e^{2iz}-1)+2i}{e^{2iz}-1}.
\end{aligned}
$$

Thus

$$\tan(\frac{\pi}{2}-z) = i - \frac{2i}{1-e^{2iz}}.$$

Now let $u = 2iz$, then

$$
\begin{aligned}
\tan(\frac{\pi}{2}-z) &= \tan(\frac{\pi}{2}-\frac{u}{2i})\\
&= i - 2i\frac{1}{1-e^{u}}\\
&= i - 2iF(e^{u}), \qquad \text{where } F(e^{u}) = \frac{1}{1-e^{u}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(\frac{d}{dz})^{n}\tan(\frac{\pi}{2}-z) &= (\frac{du}{dz}\frac{d}{du})^{n}\tan(\frac{\pi}{2}-\frac{u}{2i})\\
&= (2i)^{n}(\frac{d}{du})^{n}\tan(\frac{\pi}{2}-\frac{u}{2i})\\
&= (2i)^{n}(\frac{d}{du})^{n}(i-2iF(e^{u}))\\
&= -(2i)^{n+1}(\frac{d}{du})^{n}F(e^{u})\\
&= -(2i)^{n+1}(1-e^{u})^{-1}P_n(t)
\end{aligned}
$$

where $P_n(t) = \sum_{k=1}^{n} k!S_k^n\, t^k$, and $t = \frac{e^u}{1-e^u}$, by lemma (4.1.1).

Therefore, the nth derivative of $\tan(\frac{\pi}{2}-z)$ vanishes when the polynomial $P_n(t)$ does. But, by theorem (3.1.1), the zeros of $P_n(t)$ are simple, real and except for

$t = 0$, contained in the interval $(-1, 0)$. Thus by lemma (4.1.2),

$$u \in \{u = x + iy : y = (2k-1)\pi\}, \text{ i.e., } z \in \{z = \frac{1}{2}y - \frac{i}{2}x : y = (2k-1)\pi\}$$

Hence, the zeros of the successive derivatives of $\tan(\frac{\pi}{2} - z)$ are of the form

$$z = \frac{1}{2}(2k-1)\pi - \frac{i}{2}x = k\pi - \frac{\pi}{2} - \frac{i}{2}x.$$

Now, let $w = \frac{\pi}{2} - z$, then the zeros of the nth derivatives of $\tan w$ are of the form $w = m\pi + \frac{i}{2}x$ where $m = 0, \pm 1, \pm 2, \cdots$. Hence the real part of w is an integer multiple of π.

4.2 The Zeros of the Successive Derivatives of $f(z) = e^{-e^z}$.

Lemma 4.2.1 *Let* $P_n(t) = \sum_{k=1}^{n} S_k^n \, t^k$ *where* $t = -e^z$. *Then*

$$(\frac{d}{dz})^n(e^{-e^z}) = e^t P_n(t) \; .$$

Proof: Let $F(u) = e^{-u}$,

Then $F^{(k)}(u) = (-1)^k e^{-u}$. Thus, by proposition (4.1.1),

$$\begin{aligned}
(\frac{d}{dz})^n F(e^z) &= \sum_{k=1}^{n} S_k^n F^{(k)}(e^z) e^{kz} \\
&= \sum_{k=1}^{n} S_k^n \, (-1)^k e^{-e^z} e^{kz} \\
&= e^{-e^z} \sum_{k=1}^{n} (-1)^k S_k^n \, e^{kz} \\
&= e^{-e^z} \sum_{k=1}^{n} S_k^n t^k \text{ , where } t = -e^z.
\end{aligned}$$

Thus $(\frac{d}{dz})^n(e^{-e^z}) = e^t P_n(t)$, where $P_n(t) = \sum_{k=1}^{n} S_k^n t^k$ and $t = -e^z$.

Lemma 4.2.2 *Let $t = -e^z$. Then t is real and negative if and only if*

$$z \in \{z = x + iy : y = 2k\pi, k = 0, \pm 1, \pm 2, \cdots\} .$$

Proof:

$$\begin{aligned}
t \text{ is real} \quad &\Leftrightarrow \quad \bar{t} = t \\
&\Leftrightarrow \quad e^{\bar{z}} = e^{z} \\
&\Leftrightarrow \quad e^{2yi} = 1 \\
&\Leftrightarrow \quad y = k\pi, k = 0, \pm 1, \pm 2, \cdots \\
&\Leftrightarrow \quad -e^z = -e^{x+iy} = \begin{cases} -e^x & \text{if } k \text{ is even} \\ +e^x & \text{if } k \text{ is odd} . \end{cases}
\end{aligned}$$

Thus t is real and negative if and only if $t = -e^x$. That is, t is real and negative if and only if $z \in \{z = x + iy : y = 2k\pi, k = 0, \pm 1, \pm 2, \cdots\}$.

Theorem 4.2.1 *The zeros of the nth derivative of e^{-e^z} are contained in the set $\{z = x + iy : y = 2k\pi, k = 0, \pm 1, \pm 2, \cdots\}$.*

Proof: Lemma (4.2.1) also tells us that the zeros of the *nth* derivative of $f(z) = e^{-e^z}$ are given by $t = -e^z$ where t is a zero of the polynomial $P_n(t) = \sum_{k=1}^{n} S_k^n t^k$, which is real, simple and nonpositive. Thus by lemma (4.2.2)
$z \in \{z = x + iy : y = 2k\pi, k = 0, \pm 1, \pm 2, \cdots\}$.
Therefore the zeros of the *nth* derivative of e^{-e^z} are contained in the set

$$\{z = x + iy : y = 2k\pi, k = 0, \pm 1, \pm 2, \cdots\} .$$

4.3 The Zeros of the Successive Derivatives of $f(z) = e^{-z^2}$.

Lemma 4.3.1 *The nth derivative of $f(z) = e^{-z^2}$ is of the form*

$$f^{(n)}(z) = (-1)^n H_n(z) e^{-z^2}.$$

where

$$H_0(z) = 1, H_1(z) = 2z \quad \textit{and} \quad H_n(z) = 2zH_{n-1}(z) - H'_{n-1}(z) \ .$$

Proof: We shall use mathematical induction to show this lemma.

$$\begin{aligned} \text{For } n = 1, f'(z) &= -2ze^{-z^2} \\ &= -H_1(z)e^{-z^2}. \end{aligned}$$

Thus the formula is true for $n = 1$.

Assume now that it is true for $(n-1)$, i.e.,
$f^{(n-1)}(z) = (-1)^{n-1}H_{n-1}e^{-z^2}$. If we differentiate this equation once , we get

$$\begin{aligned} f^{(n)}(z) &= (-1)^{n-1}H'_{n-1}(z)e^{-z^2} + (-1)^n 2ze^{-z^2}H_{n-1}(z) \\ &= (-1)^n \left[2zH_{n-1}(z) - H'_{n-1}(z)\right] e^{-z^2} \\ &= (-1)^n H_n(z)e^{-z^2}. \end{aligned}$$

Lemma 4.3.2 *The polynomials $H_n(z)$ satisfying the following,*

$$H_n(z) = 2zH_{n-1}(z) - H'_{n-1}(z) \ ,$$

and $H_0(z) = 1$ and $H_1(z) = 2z$ are the Hermite polynomials.

Proof: We shall use mathematical induction to prove this lemma.

For $n = 1, H_n(z) = 2z$ and thus $H'_n(z) = 2$ and $H''_n(z) = 0$. To show that $H_1(z)$ is a Hermite polynomial we first show that it satisfies the Hermite differential equation

$$y'' - 2zy' + 2y = 0.$$

Indeed

$$H''_1(z) - 2zH'_1 + 2H_1(z) = 0 - 2z(2) + 2(2z) = 0.$$

Next we assume that $H_{n-1}(z)$ satisfies the Hermite differential equation and show that $H_n(z)$ does too. Since $H_n(z) = 2zH_{n-1}(z) - H'_{n-1}(z)$,
then $H'_n(z) = 2H_{n-1}(z) + 2zH'_{n-1}(z) - H''_{n-1}(z)$. If we add

$$2(n-1)H_{n-1}(z) - 2(n-1)H_{n-1}(z)$$

to the right hand side of this equation , then

$$\begin{aligned}
H'_n(z) &= 2H_{n-1}(z) + 2zH'_{n-1}(z) - H''_{n-1}(z) + 2(n-1)H_{n-1}(z) - 2(n-1)H_{n-1}(z) \\
&= 2H_{n-1}(z) + 2(n-1)H_{n-1}(z) \\
&= 2nH_{n-1}(z).
\end{aligned}$$

Now, we differentiate $H'_n(z)$ once again to get $H''_n(z) = 2nH'_{n-1}(z)$.

Therefore

$$\begin{aligned}
H''_n(z) - 2zH'_n(z) + 2nH_n(z) &= 2nH'_{n-1}(z) - 2z(2nH_{n-1}(z)) \\
&+ 2n(2zH_{n-1}(z) - H'_{n-1}(z)) \\
&= 2nH'_{n-1}(z) - 4nzH_{n-1}(z) + 4nzH_{n-1}(z) - 2nH'_{n-1}(z) \\
&= 0.
\end{aligned}$$

Hence $H_n(z)$ satisfies the Hermite differential equation.

Now let $H_n(z)$ and $H_m(z)$ be two polynomials satisfying the hypotheis of the lemma. Thus they satisfy the Hermite differential equation. Therefore

$$H''_n(z) - 2zH'_n(z) + 2nH_n(z) = 0$$

and

$$H''_m(z) - 2zH'_m(z) + 2mH_m(z) = 0.$$

Multiplying each of the above equations by e^{-z^2} we get

$$(e^{-z^2}H_n'(z))' + 2ne^{-z^2}H_n(z) = 0$$

and

$$(e^{-z^2}H_m'(z))' + 2me^{-z^2}H_m(z) = 0.$$

Multiplying the first equation by $H_m(z)$ and then second by $H_n(z)$ we get

$$H_m(z)(e^{-z^2}H_n'(z))' = -2ne^{-z^2}H_m(z)H_n(z)$$

and

$$H_n(z)(e^{-z^2}H_m'(z))' = -2me^{-z^2}H_m(z)H_n(z)$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-z^2}H_m(z)H_n(z)(2m-2n)dz &= \int_{-\infty}^{\infty}\left[H_m(z)(e^{-z^2}H_n'(z))' - H_n(z)(e^{-z^2}H_m'(z))'\right]dz \\
&= H_m(z)(e^{-z^2})H_n'(z)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_m'(z)(e^{-z^2}H_n'(z))dz \\
&- H_n(z)(e^{-z^2}H_m'(z))|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} H_n'(z)(e^{-z^2}H_m'(z))dz \\
&= 0.
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} e^{-z^2}H_m(z)H_n(z)dz = 0 \qquad \text{if } n \neq m.$$

Thus H_n is a Hermite polynomial by definition (3.3.1).

Theorem 4.3.1 *The zeros of the successive derivatives of $f(z) = e^{-z^2}$ are real and simple.*

Proof: Since $f^{(n)}(z) = (-1)^n H_n(z)e^{-z^2}$, by lemma (4.3.1), then the zeros of $f^{(n)}(z)$ are the same as those of $H_n(z)$. Hence, by theorem (3.3.1), the zeros of the successive derivatives of $f(z)$ are real and simple.

At this stage one will naturally ask what the final sets of the functions studied in this chapter are. The answer to this question is so easy for the function $(a - be^z)^{-1}$ since it is a meromorphic function and since we have already had a theorem concerning the final sets of meromorphic function in Chpater 2. Whereas for the case of the other two functions e^{-e^z} and e^{-z^2}, the problem is completely different. We can see that these two function are entire and we still, upto now, don't have any theorems or results that could tell us what the final sets of entire functions are. Although this is the case, the final sets of these two functions have been successfully found by some great mathematicians. The final set of the function e^{-e^z} was first suggested by George Pólya and found by Edrei to be the set of infinite parallel lines dividing the plane into congruent strips of width 2π [10;182]. While the final set of e^{-z^2} was found by George Pólya to be the real axis. [10;398]

Chapter 5

Pólya's Result On Analytic Functions And Some Recent Developments

Let U_0 be the class of entire functions f of the form

$$f(z) = cz^m e^{-\gamma z^2 + \alpha z} \Pi_{n=1}^{\infty}(1 - \frac{z}{z_n})e^{z/z_n}$$

where c is a constant, $\gamma \geq 0, \alpha$ and the z_n are real, and $\sum |z_n|^{-2} < +\infty$.

Classical theorems of Laguerre and Pólya [6;228] assert that $f \in U_0$ if and only if f can be uniformly approximated on disks about the origin by a sequence of polynomials with only real zeros. A corollary of this result is that $f \in U_0$ implies $f^{(n)} \in U_0$ for $n = 1, 2, 3, \cdots$. In particular, $f \in U_0$ implies $f^{(n)}$ has only real zeros.

In 1915, Pólya [9] asked whether the following converse assertion holds: If a (constant multiple of a) real entire function f (i.e., f takes the real axis onto the real axis) and each of its derivatives $f^{(n)}, n = 1, 2, 3, \cdots$ have only real zeros, is $f \in U_0$?

Pólya showed that if $f(z) = g(z)e^{H(z)}$ where g and H are polynomials, then the answer to this question is affirmitive.

In this chapter we shall give Pólya's proof of the above result. We shall also state without proof the result of Hellerstein and Williamson which answers the question of Pólya.

5.1 The Order and Genus of an Entire Function.

Throughout this chapter we shall use the terms "order" and "genus" of a function, so it seems to be necessary to recall definitions and some examples of these two notions.

Definition 5.1.1 *Let f be an entire function. For $r \geq 0$, put $M(r) = \sup_{|z|=r} |f(z)|$. Then $M(r)$ is called the maximum modulus of f.*

Example 5.1.1 *Let $f(z) = cz$ for some constant c.*

Then $|f(z)| = |c||z|$. Thus

$$M(r) = \sup_{|z|=r} |f(z)| = \sup_{|z|=r} |c||z| = \sup_{|z|=r} |c|r = cr, \quad r \geq 0$$

That is $M(r; cz) = cr, \quad r \geq 0$.

Example 5.1.2 *Let $a_1, \cdots, a_n$ be positive real numbers and let*

$$P(z) = (1 + \tfrac{z}{a_1}) \cdots (1 + \tfrac{z}{a_n}) .$$

Then

$$\begin{aligned} |P(z)| &= |1 + \frac{z}{a_1}| \cdots |1 + \frac{z}{a_n}| \\ &\leq (1 + \frac{|z|}{a_1}) \cdots (1 + \frac{|z|}{a_n}) \end{aligned}$$

Thus $M(r) = \sup_{|z|=r} |P(z)| = (1 + \dfrac{r}{a_1}) \cdots (1 + \dfrac{r}{a_n}), \quad r \geq 0.$

That is, $M(r; P(z)) = P(r), \quad r \geq 0.$

Example 5.1.3 *Let* $f(z) = e^z$. *Then*

$$|f(z)| = |e^z| = e^{Re(z)} = e^{r\cos(\arg z)} \leq e^r.$$

Thus $M(r) \;=\; \sup_{|z|=r} |f(z)| \;= e^r$.
That is $M(r; e^z) = e^r$.

Definition 5.1.2 *Let f be a non-constant entire function. The number λ defined by*

$$\lambda = \lim_{r\to\infty} \sup \frac{\log\log M(r;f)}{\log r}, \qquad r \geq 0,$$

is called the order of f.

Example 5.1.4 *Let*

$$P(z) = (1 + \tfrac{z}{a_1}) \cdots (1 + \tfrac{z}{a_n}) \;, a_j > 0.$$

By example 5.1.2 $M(r; P(z)) = P(r)$. *Thus*

$$\lambda = \lim_{r\to\infty} \sup \frac{\log\log P(r)}{\log r}.$$

Let $a_k = \min\{a_i\}$. *Then*

$$(1 + \frac{r}{a_1}) \cdots (1 + \frac{r}{a_n}) \leq (1 + \frac{r}{a_k})^n.$$

Thus $\log P(r) \leq n \log(1 + \frac{r}{a_k})$. *Again*
$\log\log(P(r)) \leq \log n + \log\log(1 + \frac{r}{a_k})$. *Thus*

$$\frac{\log\log(P(r))}{\log r} \leq \frac{\log n}{\log r} + \frac{\log\log(1 + \frac{r}{a_k})}{\log r}.$$

Hence

$$\lim_{r\to\infty} \sup \frac{\log\log(P(r))}{\log r} \leq 0.$$

On the other hand, let $a_\ell = \max\{a_i\}$*, then*

$$(1+\frac{r}{a_1})\cdots(1+\frac{r}{a_n}) \geq (1+\frac{r}{a_\ell})^n.$$

Thus $\log P(r) \geq n\log(1+\frac{r}{a_\ell})$*. Again*

$\log\log(P(r)) \geq \log n + \log\log(1+\frac{r}{a_\ell})$*. Thus*

$$\frac{\log\log(P(r))}{\log r} \geq \frac{\log n}{\log r} + \frac{\log\log(1+\frac{r}{a_\ell})}{\log r}.$$

Hence

$$\lim_{r\to\infty}\sup\frac{\log\log(P(r))}{\log r} \geq 0.$$

Therefore , the order of $P(z)$ *is zero.*

Example 5.1.5 *Let* $f(z) = e^z$*. Then by example (5.1.3)*

$M(r;e^z) = e^r$*. Thus*

$$\lambda = \frac{\log\log M(r;e^z)}{\log r} = \frac{\log r}{\log r} = 1.$$

Hence the order of e^z *is 1.*

Example 5.1.6 *In general, if* $f(z) = e^{z^\alpha}$ *for any positive integer* α *then the order of* $f(z)$ *is* α*.*

Definition 5.1.3 *Let* f *be an entire function, then* f *is said to be of rank* $p < \infty$ *if there is an integer* $p \geq 0$ *such that*

$$\sum_{n=1}^{\infty}|a_n|^{-(p+1)} < \infty$$

and p *is the smallest integer such that this occurs, where* $\{a_1, a_2, \cdots\}$ *are the zeros of* f *and* $|a_1| \leq |a_2| \leq \cdots$*. [2; 282]*

Definition 5.1.4 *An entire function f is said to have a finite genus if f has the form*

$$f(z) = z^m e^{H(z)} \prod_{n=1}^{\infty} (1 - \tfrac{z}{a_n}) e^{(\frac{z}{a_n} + \frac{z^2}{2a_n^2} + \cdots + \frac{z^p}{pa_n^p})}$$

where $H(z)$ is a polynomial of degree q and p is the rank of f and $\{a_1, a_2, ...\}$ are the zeros of f and $|a_1| \leq |a_2| \leq \cdots$. If $\mu = \max(p, q)$ then μ is the genus of f. [2; 283]

5.2 The Zeros of Some Derivatives of $G(z) = g(z)e^{H(z)}$

Theorem 5.2.1 *Let $G(z) = g(z)e^{H(z)}$, where g and H are polynomials, and H is of degree $(m+1)$. Then*

(i) If $(m+1) \geq 3$, G always has some derivatives having non-real zeros.

(ii) If $(m+1) = 2$ and the coefficient of z^2 in $H(z)$ is positive, G always has some derivatives having non-real zeros. [9;125]

To prove this theorem, we shall use the following lemmas.

Lemma 5.2.1 *Let $G(z) = g(z)e^{H(z)}$, where g and H are polynomials and let $h(z) = H'(z)$. Then*

$$G^{(n)}(z) = g_n(z)e^{H(z)}$$

where $g_{n+1}(z) = g_n h + g_n'$ and $g_0(z) = g(z)$.

Proof: We shall use mathematical induction to prove this lemma. For $n = 1$,

$$\begin{aligned} G^{(1)}(z) &= g'(z)e^{H(z)} + H'(z)g(z)e^{H(z)} \\ &= [g'(z) + h(z)g(z)]e^{H(z)} \\ G^{(1)}(z) &= g_1(z)e^{H(z)}. \end{aligned}$$

Thus the lemma is true for $n = 1$.

Now suppose that it is also true for $(n-1)$, i.e., $G^{(n-1)}(z) = g_{n-1}(z)e^{H(z)}$. Then, if we differentiate both sides once, we get

$$\begin{aligned} G^{(n)}(z) &= g'_{n-1}(z)e^{H(z)} + H'(z)g_{n-1}(z)e^{H(z)} \\ &= [g'_{n-1}(z) + h(z)g_{n-1}(z)]e^{H(z)}. \end{aligned}$$

Thus $G^{(n)}(z) = g_n(z)e^{H(z)}$ as required.

Lemma 5.2.2 *Let $g_{n+1}(z) = g_n h + g'_n$, then*

$$\frac{g'_{n+1}}{g_{n+1}} = \frac{g'_n}{g_n} + \frac{h'}{h} + \frac{1}{1+\frac{g'_n}{hg_n}}\frac{d}{dz}(\frac{g'_n}{hg_n}) .$$

Proof: Since $g_{n+1}(z) = g_n(z)h(z) + g'_n(z)$, then

$$\begin{aligned} \frac{g'_{n+1}}{g_{n+1}} &= \frac{g''_n + (gh_n)'}{g'_n + gh_n} \\ &= \frac{g''_n}{g'_n + gh_n} + \frac{(gh_n)'}{g'_n + gh_n} \\ &= \frac{g''_n}{g'_n + hg_n} + \frac{(hg_n)'}{hg_n}(\frac{hg_n}{g'_n + hg_n}) \\ &= \frac{g''_n}{g'_n + gh_n} + \left[\left(\frac{h'g_n}{hg_n} + \frac{hg'_n}{hg_n}\right)\left(1 - \frac{g'_n}{g'_n + hg_n}\right)\right] \\ &= \frac{g''_n}{g'_n + hg_n} + \left[\left(\frac{h'}{h} + \frac{g'_n}{g_n}\right) - \frac{g'_n(hg_n)'}{(g'_n + hg_n)hg_n}\right] \\ &= \frac{g'_n}{g_n} + \frac{h'}{h} + \frac{g''_n}{g'_n + hg_n} - \frac{g'_n(hg_n)'}{(g'_n + hg_n)hg_n} \\ &= \frac{g'_n}{g_n} + \frac{h'}{h} + \frac{g''_n hg_n - g'_n(hg_n)'}{(g'_n + hg_n)hg_n} \\ &= \frac{g'_n}{g_n} + \frac{h'}{h} + \left(\frac{hg_n}{g'_n + hg_n}\right)\left(\frac{g''_n hg_n - g'_n(hg_n)'}{(hg_n)^2}\right). \end{aligned}$$

Thus

$$\frac{g'_{n+1}}{g_{n+1}} = \frac{g'_n}{g_n} + \frac{h'}{h} + \frac{1}{1+\frac{g'_n}{hg_n}}\frac{d}{dz}(\frac{g'_n}{hg_n}) .$$

Another Proof: Another easier verification of the above formula can be done as follows:

Since $g_{n+1} = hg_n + g_n'$,

then $g_{n+1} = hg_n(1+\frac{g_n'}{hg_n})$, and note that $hg_n \equiv 0$ since neither h nor g_n is identically zero, otherwise H will be a constant or $G^{(n-1)}(z)$ is a constant.

So logarithmic differentiation gives

$$\frac{g_{n+1}'}{g_{n+1}} = \frac{g_n'}{g_n} + \frac{h'}{h} + \frac{1}{1+\frac{g_n'}{hg_n}}\frac{d}{dz}(1+\frac{g_n'}{hg_n}) = \frac{g_n'}{g_n} + \frac{h'}{h} + \frac{1}{1+\frac{g_n'}{hg_n}}\frac{d}{dz}(\frac{g_n'}{hg_n}).$$

Lemma 5.2.3 *(i) If $r_{n_1}, \cdots, r_{n_k}$ are the roots (repeated with multiplicity) of $g_n(z)$ and*

$$S_{n,v} = \sum_{i=1}^{k}(r_{n_i})^v, \text{ then}$$
$$\frac{g_n'(z)}{g_n(z)} = \sum_{v=0}^{\infty}\frac{S_{n,v}}{z^{v+1}}.$$

(ii) If $r_1, \cdots, r_l$ are the roots (repeated with multiplicity) of $g(z)$ and

$$s_v = \sum_{i=1}^{l}(r_i)^v \text{ , then}$$

$$\frac{g'(z)}{g(z)} = \sum_{v=0}^{\infty}\frac{s_v}{z^{v+1}}.$$

(iii) If $c_1, \cdots, c_m$ are the roots (repeated with multiplicity) of $h(z)$ and

$$t_v = \sum_{i=1}^{m}(c_i)^v \text{ , then}$$

$$\begin{aligned} \frac{h'}{h} &= \sum_{v=0}^{\infty}\frac{t_v}{z^{v+1}}, && \text{and} \\ \frac{1}{h} &= \sum_{v=0}^{\infty}\frac{P_v}{z^{m+v}}, && \text{where } P_v = \sum_{i=1}^{m}\frac{a_i c_i^v}{b} \end{aligned}$$

for some constants $a_1, \cdots, a_m$ and b.

Proof:

(i) Since $r_{n_1}, \cdots, r_{n_k}$ are the roots of $g_n(z)$ then we can write $g_n(z)$ as follows:

$$g_n(z) = a_n(z - r_{n_1}) \cdots (z - r_{n_k})$$

where $a_n \neq 0$ is a constant. So

$$\frac{g_n'(z)}{g_n(z)} = \frac{1}{z - r_{n_1}} + \cdots + \frac{1}{z - r_{n_k}}.$$

But

$$\begin{aligned}
\frac{1}{z - r_{n_i}} &= \frac{1}{z(1 - \frac{r_{n_i}}{z})}, && z \neq 0 \\
&= \frac{1}{z} \sum_{v=0}^{\infty} \left(\frac{r_{n_i}}{z}\right)^v, && \text{if } |z| > |r_{n_i}| \\
&= \sum_{v=0}^{\infty} \frac{(r_{n_i})^v}{z^{v+1}}.
\end{aligned}$$

So

$$\begin{aligned}
\frac{g_n'(z)}{g_n(z)} &= \sum_{i=1}^{k} \frac{1}{z - r_{n_i}} \\
&= \sum_{i=1}^{k} \left(\sum_{v=0}^{\infty} \frac{(r_{n_i})^v}{z^{v+1}} \right) \\
&= \sum_{v=0}^{\infty} \left(\frac{1}{z^{v+1}} \sum_{i=1}^{k} (r_{n_i})^v \right).
\end{aligned}$$

Thus $\frac{g_n'(z)}{g_n(z)} = \sum_{v=0}^{\infty} \frac{S_{n,v}}{z^{v+1}}$, provided $|z| > \max_i(|r_{n_i}|)$, as required.

(ii) Using the same argument, we can show that

$$\frac{g'}{g} = \sum_{v=0}^{\infty} \frac{s_v}{z^{v+1}}.$$

(iii) Again a similar argument will show that

$$\frac{h'}{h} = \sum_{v=0}^{\infty} \frac{t_v}{z^{v+1}} .$$

Finally, we show that

$$\frac{1}{h(z)} = \sum_{v=0}^{\infty} \frac{P_v}{z^{m+v}} .$$

Since $c_1, \cdots, c_m$ are the roots of $h(z)$, then

$$\begin{aligned} h(z) &= b(z-c_1)\cdots(z-c_m) \qquad \text{for some constant } b \neq 0 \\ &= bz^m(1-\frac{c_1}{z})\cdots(1-\frac{c_m}{z}), \qquad z \neq 0. \end{aligned}$$

Thus

$$\frac{1}{h(z)} = \frac{1}{bz^m} \frac{1}{(1-\frac{c_1}{z})\cdots(1-\frac{c_m}{z})}$$

so, the partial fraction decomposition gives,

$$\frac{1}{h(z)} = \frac{1}{bz^m}\left\{\frac{a_1}{(1-\frac{c_1}{z})} + \cdots + \frac{a_m}{(1-\frac{c_m}{z})}\right\},$$

for some constants $a_1, \cdots, a_m$.

But

$$\frac{a_i}{1-\frac{c_i}{z}} = a_i \sum_{v=0}^{\infty} (\frac{c_i}{z})^v,$$

therefore

$$\begin{aligned} \frac{1}{h(z)} &= \frac{1}{bz^m} \sum_{i=1}^{m} \left[a_i \sum_{v=0}^{\infty} (\frac{c_i}{z})^v\right] \\ &= \frac{1}{bz^m} \sum_{v=0}^{\infty} \left[\frac{\sum_{i=1}^{m} a_i c_i^v}{z^v}\right] \end{aligned}$$

or

$$\frac{1}{h(z)} = \sum_{v=0}^{\infty} \frac{P_v}{z^{m+v}} \quad \text{where}$$

$$P_v = \sum_{i=1}^{m} \frac{a_i c_i^v}{b} \text{ and } |z| > \max_i(|c_i|),$$

as desired.

Lemma 5.2.4 *Let $S_{n,v}, s_v, t_v$ and P_k as in lemma (5.2.3). Then*

$$S_{n,v} = s_v + nt_v, \qquad \textit{for } v = 1, 2, \cdots, m \qquad \textit{and}$$

$$S_{n,v} = s_v + nt_v - v \sum_{k=0}^{v-(m+1)} P_k \left(ns_{v-(m+1)-k} + \binom{n}{2} t_{v-(m+1)-k} \right),$$

for $v = m+1, m+2, \cdots, 2m+2$.

Proof : First we try to equate the coefficients of the terms having the same power of z in both sides of the equation

$$\frac{g'_{n+1}}{g_{n+1}} = \frac{g'_n}{g_n} + \frac{h'}{h} + \frac{1}{1 + \frac{g'_n}{hg_n}} \frac{d}{dz}\left(\frac{g'_n}{hg_n}\right) \tag{5.1}$$

using the expansions found in lemma (5.2.3).

Since

$$\frac{g'_n}{g_n} = \sum_{v=0}^{\infty} \frac{S_{n,v}}{z^{v+1}}, \text{ and}$$

$$\frac{1}{h} = \sum_{v=0}^{\infty} \frac{P_v}{z^{m+v}}$$

then

$$\begin{aligned}
\frac{g'_n}{hg_n} &= \left(\sum_{v=0}^{\infty} \frac{S_{n,v}}{z^{v+1}} \right) \left(\sum_{v=0}^{\infty} \frac{P_v}{z^{m+v}} \right) \\
&= \frac{P_0 S_{n,0}}{z^{m+1}} + \frac{P_1 S_{n,0} + P_0 S_{n,1}}{z^{m+2}} \\
&+ \frac{P_2 S_{n,0} + P_1 S_{n,1} + P_0 S_{n,2}}{z^{m+3}} + \cdots .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{d}{dz}\left(\frac{g'_n}{g_n h}\right) &= -(m+1)\frac{P_0 S_{n,0}}{z^{m+2}} - (m+2)\frac{P_1 S_{n,0} + P_0 S_{n,1}}{z^{m+3}} \\
&- (m+3)\frac{P_2 S_{n,0} + P_1 S_{n,1} + P_0 S_{n,2}}{z^{m+4}} - \cdots .
\end{aligned}$$

Also

$$\frac{g'_{n+1}}{g_{n+1}} = \sum_{v=0}^{\infty} \frac{S_{n+1,v}}{z^{v+1}}.$$

Hence equation (5.1) becomes

$$
\begin{aligned}
\frac{S_{n+1,0}}{z}+\cdots+\frac{S_{n+1,m}}{z^{m+1}} \quad + \quad \cdots &= \frac{g_n'}{g_n}+\frac{h'}{h}+\frac{1}{1+\frac{g_n'}{hg_n}}\frac{d}{dz}\left(\frac{g_n'}{hg_n}\right)\\
&= \frac{g_n'}{g_n}+\frac{h'}{h}+\frac{d}{dz}\left(\frac{g_n'}{hg_n}\right)\sum_{j=0}^{\infty}(-1)^j\left(\frac{g_n'}{hg_n}\right)^j\\
&= \frac{g_n'}{g_n}+\frac{h'}{h}+\frac{d}{dz}\left(\frac{g_n'}{hg_n}\right)-\frac{g_n'}{hg_n}\frac{d}{dz}\left(\frac{g_n'}{hg_n}\right)+\left(\frac{g_n'}{hg_n}\right)^2\frac{d}{dz}\left(\frac{g_n'}{hg_n}\right)-\cdots.
\end{aligned}
$$

Since $\frac{g_n'}{g_n}$; $\frac{h'}{h}$ and $\frac{d}{dz}(\frac{g_n'}{hg_n})$ are the only terms, of the right hand side of the above equation, with expansions having terms where the powers of $(\frac{1}{z})$ are less than or equal to $(2m+2)$, then we don't have to expand the remaining terms of that equation. Thus it will become

$$
\begin{aligned}
\frac{S_{n+1,0}}{z}+\cdots \quad + \quad & \frac{S_{n+1,m}}{z^{m+1}}+\cdots = \sum_{v=0}^{\infty}\frac{S_{n,v}+t_v}{z^{v+1}}-(m+1)\frac{P_0S_{n,0}}{z^{m+2}}\\
- \quad & (m+2)\frac{P_1S_{n,0}+P_0S_{n,1}}{z^{m+3}}+\cdots\\
= \quad & \frac{S_{n,0}+t_0}{z}+\cdots+\frac{S_{n,m}+t_m}{z^{m+1}}+\frac{S_{n,m+1}+t_{m+1}-(m+1)P_0S_{n,0}}{z^{m+2}}\\
+ \quad & \frac{S_{n,m+2}+t_{m+2}-(m+2)(P_1S_{n,0}+P_0S_{n,1})}{z^{m+3}}+\cdots.
\end{aligned}
$$

We now can equate the coefficients of the terms with similar powers of z and find out that $S_{n+1,v}=S_{n,v}+t_v \qquad$ for $v=0,1,\cdots,m$,

and

$$S_{n+1,v}=S_{n,v}+t_v-v\sum_{k=0}^{v-(m+1)}P_k\,S_{n,v-(m+1)-k}$$

for $v=m+1,\cdots,2m+2$.

Hence

$$
\begin{aligned}
S_{n,v} &= S_{n-1,v}+t_v\\
&= S_{n-2,v}+t_v+t_v=S_{n-2,v}+2t_v\\
&= S_{n-3,v}+t_v+2t_v=S_{n-3,v}+3t_v
\end{aligned}
$$

$$\vdots$$

$$= \quad S_{n-n,v} + t_v + (n-1)t_v = S_{0,v} + nt_v$$

$$S_{n,v} \quad = \quad s_v + nt_v, \qquad \text{for } v = 1, \cdots, m.$$

Also

$$S_{n+1,}v = S_{n,v} + t_v - v \sum_{k=0}^{v-(m+1)} P_k \left(s_{v-(m+1)-k} + nt_{v-(m+1)-k}\right).$$

So

$$\begin{aligned} S_{n,v} &= S_{n-1,v} + t_v - v \sum_{k=0}^{v-(m+1)} P_k \left(s_{v-(m+1)-k} + (n-1)t_{v-(m+1)-k}\right) \\ &= S_{n-2,v} + 2t_v - v \sum_{k=0}^{v-(m+1)} P_k \left(2s_{v-(m+1)-k} + ((n-1)+(n-2))\, t_{v-(m+1)-k}\right) \\ &\vdots \\ &= S_{0,v} + nt_v \\ &- v \sum_{k=0}^{v-(m+1)} P_k \left(ns_{v-(m+1)-k} + ((n-1)+(n-2)+\cdots+(n-n))\, t_{v-(m+1)-k}\right). \end{aligned}$$

Thus

$$S_{n,v} = s_v + nt_v - v \sum_{k=0}^{v-(m+1)} P_k \left(ns_{v-(m+1)-k} + \binom{n}{2} t_{v-(m+1)-k}\right)$$

for $v = m+1, \cdots, 2m+2$.

Proof of Theorem 5.2.1: We shall prove this theorem by contradiction. Suppose that $G', G'', \cdots$ have only real zeros, then $g_n(z)$ has only real zeros since $G^{(n)}(z) = g_n(z)e^{H(z)}$ where $g_n(z) = g_{n-1}h + g'_{n+1}$ (lemma (5.2.1)). But if $g_n(z)$ has only real zeros then

$$\sum_{i=1}^{k} (r_{n_i})^{2v} = S_{n,2v} \geq 0 \tag{5.2}$$

where $r_{n_i}, \cdots, r_{n_k}$ are the roots of $g_n(z)$. Also

$$S^2_{n,\mu+v} \quad = \quad \left[\sum_{i=1}^{k} (r_{n_i})^{\mu+v}\right]^2$$

$$
\begin{aligned}
&= \left[\sum_{i=1}^{k} r_{n_i}^{\mu} r_{n_i}^{v}\right]^2 \\
&\le \left(\sum_{i=1}^{k} r_{n_i}^{2\mu}\right)\left(\sum_{i=1}^{k} r_{n_i}^{2v}\right) \\
&= S_{n,2\mu}S_{n,2v} .
\end{aligned}
$$

Thus

$$
\left(S_{n,2\mu}S_{n,2v} - S_{n,\mu+v}^2\right) \ge 0. \tag{5.3}
$$

Our aim now is to find a contradiction to the two inequalities (5.2.) and (5.3).

First note that for sufficiently large n the polynomial $S_{n,v}$, with respect to n , has the same sign as the sign of the coefficient of the highest power of n. Similarly for $S_{n,2v}$ and $\left(S_{n,2\mu}S_{n,2v} - S_{n,\mu+v}^2\right)$. Then consider the following two cases.

(i) For the case where $m+1 \ge 3$ and $(m+1)$ is odd. Since the highest power of n in $S_{n,m}S_{n,m+2}$ is 3 while the highest power of n in $S_{n,m+1}^2$ is 4 , then 4 is the highest power of n in $\left(S_{n,m}S_{n,m+2} - S_{n,m+1}^2\right)$. But

$$
\begin{aligned}
S_{n,m+1} &= s_{m+1} + nt_{m+1} - (m+1)P_0 n s_0 - \frac{n^2-n}{2}(m+1)t_0P_0 \\
&= -\frac{(m+1)}{2}t_0P_0n^2 + (t_{m+1} - (m+1)P_0s_0 + \frac{1}{2}(m+1)t_0P_0)n + s_{m+1}.
\end{aligned}
$$

Thus the highest power of n in $-S_{n,m+1}^2$ is $-\left(\frac{(m+1)t_0P_0}{2}\right)^2$ which is negative. Consequently, the highest power of n in $\left(S_{n,m}S_{n,m+2} - S_{n,m+1}^2\right)$ is negative which contradicts inequality (5.3).

Similarly, for $m+1 \ge 3$ and $(m+1)$ is even, we can see that the highest power of n in $\left(S_{n,m-1}S_{n,m+3} - S_{n,m+1}^2\right)$ is $-\left(\frac{(m+1)t_0P_0}{2}\right)^2$ which is again negative and thus a contradiction to inequality (5.3) .

(ii) For the case where $m+1 = 2$ and the coefficient of z^2 in $H(z)$ is positive. Since the degree of $H(z)$ is 2, the degree of $h(z) = H'(z)$ is 1. So we can write

$h(z)$ in the following form

$$\begin{aligned} h(z) &= b(z-c) \quad \text{where } b \text{ is positive} \\ &= bz(1-\frac{c}{z}). \end{aligned}$$

Hence

$$\frac{1}{h(z)} = \frac{1}{bz}\left\{\frac{1}{(1-\frac{c}{z})}\right\}$$

Since $P_v = \sum_{i=1}^{m} \frac{a_i c_i^v}{b}$ and $m = 1$, then $P_0 = \frac{a_1}{b}$. But $a_1 = 1$, thus $P_0 > 0$.
Let us now look at $S_{n,2}$,

$$\begin{aligned} S_{n,2} &= s_2 + nt_2 - 2P_0 n s_0 - (n^2 - n)t_0 P_0 \\ &= -t_0 P_0 n^2 + (t_2 - 2P_0 s_0 + t_0 P_0)n + s_2. \end{aligned}$$

Since $t_0 = \sum_{i=1}^{m}(c_i)^0 = m = 1 > 0$, then the coefficient of the highest power of n in $S_{n,2}$ is $-t_0 P_0$ which is negative. Thus a contradiction to inequality (5.2).

Corollary 5.2.1 *If $f(z) = g(z)e^{H(z)}$ where g and H are polynomials, and the zeros of f and all its derivatives are real. Then $f \in U_0$.*

Proof: By theorem (5.2.1) , $H(z)$ must be of degree less than 2 or of degree equals to 2 provided that the coefficient of z^2 in $H(z)$ is negative. Therefore $f(z)$ must belong to the class of entire functions U_0.

5.3 Some Recent Developments on the Zero Sets

In 1914, Pólya conjectured that an entire function which has, along with all its derivatives, only real zeros must have one of the following forms:

$$f(z) = Ae^{Bz} \tag{5.4}$$

$$f(z) = A(e^{icz} - e^{id}) \tag{5.5}$$

$$f(z) = Az^m e^{-az^2+bz} \Pi_{n=1}^{\infty}(1 - \frac{z}{a_n})e^{z/a_n} \tag{5.6}$$

where A and B are complex constants, $a \geq 0, b, c, d$ and $a_1, \cdots, a_n, \cdots$ are real and $\sum_{n=1}^{\infty}(\frac{1}{a_n})^2 < \infty$ and $m \geq 0$. [5; 319]

Later in 1983, Hellerstein, Shen and Williamson proved a stronger form of Pólya's conjecture. They proved that "if f is an entire function which has along with its first three derivatives, only real zeros, then f is of the form (5.4), (5.5) or (5.6) ". [5; 320]

Two more conjectures were also made by Pólya. The statements of these conjectures are:

(I) "If the order of the real entire function $f(z)$ is less than 2, and $f(z)$ has only a finite number of complex zeros, then its derivatives, from a certain one onwards, will have no complex zeros at all. " [10;182]

(II) "If the order of the real entire function $f(z)$ is greater than 2, and $f(z)$ has only a finite number of complex zeros, then the number of the complex zeros of $f^{(n)}(z)$ tends to infinity as $n \to \infty$." [10;182]

Throughout the thesis we were working on finding or trying to find the final sets of meromorphic and entire functions. It is natural to ask whether any set can be the final set of some entire functions. Since derived sets are always closed, it is necessary to start with a set which is closed. This leads to the following converse problem: If S is a given set such that $\infty \in S$ and such that the image of S on the complex sphere is compact, can we find an entire function f such that the final set of f is the set S ?

This converse problem has a very simple solution shown by Edrie and Maclane. They have shown that it is always possible to find an entire function f, having some preassigned order λ, such that the final set of f is the set S. [3]

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